Mathematics 2016 (Outside Delhi) SET I

Time allowed : 3 hours **Maximum marks : 100**

SECTION — A

1. If $x \in N$ and $\begin{vmatrix} x+3 & -2 \\ -3x & 2x \end{vmatrix} = 8$, then find the value of x. **[1]**

Solution : We have, $\begin{vmatrix} x+3 & -2 \\ -3x & 2x \end{vmatrix} = 8$

$$\Rightarrow \quad (x+3) \times 2x - (-2) \times (-3x) = 8$$

$$\Rightarrow \quad 2x^2 + 6x - 6x = 8$$

$$\Rightarrow \quad 2x^2 = 8$$

$$\Rightarrow \quad x^2 = 4$$

But $x \neq -2$ as $x \in N$

$$\Rightarrow \quad x = 2$$ **Ans.**

2. Use elementary column operation $C_2 \to C_2 + 2C_1$ in the following matrix equation :

$$\begin{bmatrix} 2 & 1 \\ 2 & 0 \end{bmatrix} = \begin{bmatrix} 3 & 1 \\ 2 & 0 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix}$$ **[1]**

Solution : We have

$$\begin{bmatrix} 2 & 1 \\ 2 & 0 \end{bmatrix} = \begin{bmatrix} 3 & 1 \\ 2 & 0 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix}$$

$$P = A.B \text{ (say)}$$

Applying $C_2 \to C_2 + 2C_1$ on P, we get

$$\begin{bmatrix} 2 & 5 \\ 2 & 4 \end{bmatrix} = Q \text{ (say)}$$

Applying $C_2 \to C_2 + 2C_1$ on B, we get

$$\begin{bmatrix} 1 & 2 \\ -1 & -1 \end{bmatrix} = C \text{ (say)}$$

Now, $$AC = \begin{bmatrix} 3 & 1 \\ 2 & 0 \end{bmatrix}\begin{bmatrix} 1 & 2 \\ -1 & -1 \end{bmatrix} = Q$$

$$\therefore \quad \begin{bmatrix} 2 & 5 \\ 2 & -4 \end{bmatrix} = \begin{bmatrix} 3 & 1 \\ 2 & 0 \end{bmatrix}\begin{bmatrix} 1 & 2 \\ -1 & -1 \end{bmatrix}$$ **Ans.**

3. Write the number of all possible matrices of order 2×2 with each entry 1, 2 or 3. **[1]**

Solution : Total number of all possible matrices of order 2×2 with each entry 1, 2 or 3 are 3^4 *i.e.*, 81. **Ans.**

4. Write the position vector of the point which divides the join of points with position vector $3\vec{a} - 2\vec{b}$ and $2\vec{a} + 3\vec{b}$ in the ratio 2 : 1. **[1]**

Solution : Let A and B be the given points with position vectors $3\vec{a} - 2\vec{b}$ and $2\vec{a} + 3\vec{b}$ respectively.

Let P and Q be the points dividing AB in the ratio 2 : 1 internally and externally respectively. Then,

$$\text{Position vector of P} = \frac{1(3\vec{a} - 2\vec{b}) + 2(2\vec{a} + 3\vec{b})}{1+2}$$

$$= \frac{7\vec{a}}{3} + \frac{4\vec{b}}{3}$$

$$\text{Position vector of Q} = \frac{1(3\vec{a} - 2\vec{b}) - 2(2\vec{a} + 3\vec{b})}{1-2}$$

$$= \vec{a} + 8\vec{b}$$ **Ans.**

5. Write the number of vectors of unit length perpendicular to both the vectors $\vec{a} = 2\hat{i} + \hat{j} + 2\hat{k}$ and $\vec{b} = \hat{j} + \hat{k}$. **[1]**

Solution : We know that the unit vectors perpendicular to the plane of $\vec{a}$ and $\vec{b}$ are $\pm \frac{\vec{a} \times \vec{b}}{|\vec{a} \times \vec{b}|}$

So, $$|\vec{a} \times \vec{b}| = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2 & 1 & 2 \\ 0 & 1 & 1 \end{vmatrix}$$

$$= (1-2)\hat{i} - (2-0)\hat{j} + (2-0)\hat{k}$$

$$= -\hat{i} - 2\hat{j} + 2\hat{k}$$

$$\Rightarrow \quad |\vec{a} \times \vec{b}| = \sqrt{(-1)^2 + (-2)^2 + (2)^2} = \sqrt{9} = 3$$

Hence, required vectors

$$= \pm \frac{1}{3}(-\hat{i} - 2\hat{j} + 2\hat{k})$$

and number of vectors are 2. **Ans.**

6. Find the vector equation of the plane with intercepts 3, – 4 and 2 on x, y and z-axis respectively. **[1]**

Solution : The equation of the required plane is,

$$\frac{x}{3} + \frac{y}{-4} + \frac{z}{2} = 1$$

$$\Rightarrow \quad 4x - 3y + 6z = 12$$

In vector form : $(x\hat{i} + y\hat{j} + z\hat{k}).(4\hat{i} - 3\hat{j} + 6\hat{k}) = 12$

i.e., $$\vec{r}.(4\hat{i} - 3\hat{j} + 6\hat{k}) = 12$$

This is the vector equation of the plane with intercept 3, – 4 and 2 on coordinate axis. **Ans.**

SECTION — B

7. Find the coordinates of the point where the line through the points A(3, 4, 1) and B(5, 1, 6) crosses the XZ-plane. Also find the angle which this line makes with the XZ-plane. **[4]**

Solution : The equation of the line passing through A and B is,

$$\frac{x-3}{5-3}=\frac{y-4}{1-4}=\frac{z-1}{6-1} \text{ or } \frac{x-3}{2}=\frac{y-4}{-3}=\frac{z-1}{5} \quad ...(i)$$

The coordinates of any point on this line are given by

$$\frac{x-3}{2}=\frac{y-4}{-3}=\frac{z-1}{5}=\lambda$$

$$\Rightarrow \quad x=2\lambda+3,\ y=-3\lambda+4,\ z=5\lambda+1$$

So $(2\lambda+3, -3\lambda+4, 5\lambda+1)$ are the coordinates of any point on the line passing through A and B. If it lies on XZ-plane then $y=0$.

$$\therefore \quad -3\lambda+4 = 0 \Rightarrow \lambda=\frac{4}{3}$$

So, the coordinates of required point are

$$\left(2\times\frac{4}{3}+3, -3\times\frac{4}{3}+4, 5\times\frac{4}{3}+1\right) i.e., \left(\frac{17}{3}, 0, \frac{23}{3}\right)$$

Now, the line in equation (i) is parallel to the vector $\vec{b}=2\hat{i}-3\hat{j}+5\hat{k}$ and the XZ-plane is normal to the vector $\vec{n}=\hat{j}$. Therefore, the angle θ between them is given by

$$\sin\theta = \left|\frac{\vec{b}\cdot\vec{n}}{|\vec{b}|\cdot|\vec{n}|}\right|$$

$$= \left|\frac{(2\hat{i}-3\hat{j}+5\hat{k})(\hat{j})}{\sqrt{(2)^2+(-3)^2+(5)^2}\sqrt{(1)^2}}\right|$$

$$\Rightarrow \quad \sin\theta = \frac{3}{\sqrt{38}}$$

$$\Rightarrow \quad \theta = \sin^{-1}\left(\frac{3}{\sqrt{38}}\right)$$

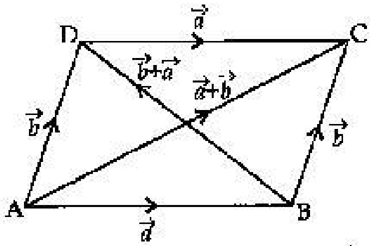
$$\Rightarrow \quad \theta = \sin^{-1}(0.4866)$$ **Ans.**

8. The two adjacent sides of a parallelogram are $2\hat{i}-4\hat{j}-5\hat{k}$ and $2\hat{i}+2\hat{j}+3\hat{k}$. Find the two unit vectors parallel to its diagonals. Using the diagonal vectors, find the area of the parallelogram. **[4]**

Solution : Let ABCD be a parallelogram such that

$$\vec{AB} = \vec{a}=2\hat{i}-4\hat{j}-5\hat{k}$$

and $$\vec{BC} = \vec{b}=2\hat{i}+2\hat{j}+3\hat{k}$$



Then,

$$\vec{AB}+\vec{BC} = \vec{AC}$$

$$\Rightarrow \quad \vec{AC} = \vec{a}+\vec{b} = 4\hat{i}-2\hat{j}-2\hat{k}$$

and $$\vec{AB}+\vec{BD} = \vec{AD}$$

$$\Rightarrow \quad \vec{BD} = \vec{AD}-\vec{AB}$$

$$\Rightarrow \quad \vec{BD} = \vec{b}-\vec{a} = 0\hat{i}+6\hat{j}+8\hat{k}$$

Now, $$\vec{AC} = 4\hat{i}-2\hat{j}-2\hat{k}$$

$$|\vec{AC}| = \sqrt{(4)^2+(-2)^2+(-2)^2} = \sqrt{24} = 2\sqrt{6}$$

and $$|\vec{BD}| = 0\hat{i}+6\hat{j}+8\hat{k}$$

$$|\vec{BD}| = \sqrt{(0)^2+(6)^2+(8)^2} = \sqrt{100} = 10$$

Unit vector along $\vec{AC}$

$$= \frac{\vec{AC}}{|\vec{AC}|} = \frac{1}{2\sqrt{6}}(4\hat{i}-2\hat{j}-2\hat{k})$$

$$= \frac{1}{\sqrt{6}}(2\hat{i}-\hat{j}-\hat{k})$$

Unit vector along $\vec{BD}$

$$= \frac{\vec{BD}}{|\vec{BD}|} = \frac{1}{10}(6\hat{j}+8\hat{k}) = \frac{1}{5}(3\hat{j}+4\hat{k})$$

Now, area of parallelogram

$$= \frac{1}{2}|\vec{AC}\times\vec{BD}|$$

$$\Rightarrow \quad \vec{AC}\times\vec{BD} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 4 & -2 & -2 \\ 0 & 6 & 8 \end{vmatrix}$$

$$= 4\hat{i}+32\hat{j}+24\hat{k}$$

Area of parallelogram

$$= \frac{1}{2}\left|\vec{AC}\times\vec{BD}\right|$$

$$= \sqrt{404} \text{ or } 2\sqrt{101} \text{ sq. units}$$ **Ans.**

9. In a game, a man wins ₹ 5 for getting a number greater than 4 and loses ₹ 1 otherwise, when a fair die is thrown. The man decided to throw

a die thrice but to quit as and when he gets a number greater than 4. Find the expected value of the amount he wins/loses. [4]

Solution : Let n denote the number of throws required to get a number greater than 4 and X denote the amount won/lost.

The man may get a number greater than 4 in the very first throw of the die or in second throw or in the third throw.

Let p = Probability of getting a number greater than 4

$$= \frac{2}{6}$$

$$q = 1-p = \frac{4}{6}$$

Thus, we have the following probability distribution for X.

Number of throws (n)	1	2	3	3
Amount won/lost (X)	5	4	3	-3
Probability (P(X))	$\frac{2}{6}$	$\frac{4}{6}\times\frac{2}{6}$	$\frac{4}{6}\times\frac{4}{6}\times\frac{2}{6}$	$\frac{4}{6}\times\frac{4}{6}\times\frac{4}{6}$

$$E(X) = 5\times\frac{2}{6}+4\times\frac{4}{6}\times\frac{2}{6}+3\times\frac{4}{6}\times\frac{4}{6}\times\frac{2}{6}+(-3)\times\frac{4}{6}\times\frac{4}{6}\times\frac{4}{6}$$

$$= \frac{19}{9}$$

Expected amount he could wins is ₹ $\frac{19}{9}$ **Ans.**

OR

A bag contains 4 balls. Two balls are drawn at random (without replacement) and are found to be white. What is the probability that all balls in the bag are white ?

Solution : We know that the number of white balls can't be less than 2.

Now, there are different cases, for the number of white balls in the bag. The total cases are ${}^2C_2$ + ${}^3C_2 + {}^4C_2$.

$$= \frac{2!}{2!\times 0!}+\frac{3!}{2!\times 1!}+\frac{4!}{2!\times 2!}$$

$$= 1+3+6 = 10$$

$\therefore$ Probability of the case that there are 4 white balls

i.e., ${}^4C_2 = 6$

Hence, the probability that all balls in the bag are white is

$$\frac{6}{10} \text{ or } \frac{3}{5}$$ **Ans.**

10. Differentiate $x^{\sin x}+(\sin x)^{\cos x}$ with respect to x. [4]

Solution : We have,

$$x^{\sin x}+(\sin x)^{\cos x}$$

Let $y = x^{\sin x}+(\sin x)^{\cos x}$

Taking log on both sides,

$$\log y = \sin x\log x+\cos x\log(\sin x)$$

$$\Rightarrow \quad y = e^{\sin x\log x}+e^{\cos x\log(\sin x)}$$

On differentiating both sides w.r.t. x, we get

$$\frac{dy}{dx} = e^{\sin x\log x}\frac{d}{dx}(\sin x\log x)+e^{\cos x\log(\sin x)}\frac{d}{dx}(\cos x\log(\sin x))$$

$$\Rightarrow \frac{dy}{dx} = x^{\sin x}\left\{\cos x\log x+\frac{\sin x}{x}\right\}+(\sin x)^{\cos x}\left\{-\sin x\log(\sin x)+\cos x.\frac{1}{\sin x}.\cos x\right\}$$

$[\because e^{\sin x\log x} = x^{\sin x}$ and $e^{\cos x\log(\sin x)} = (\sin x)^{\cos x}]$

$$= x^{\sin x}\left\{\cos x\log x+\frac{\sin x}{x}\right\}+(\sin x)^{\cos x}\left\{-\sin x\log(\sin x)+\frac{\cos^2 x}{\sin x}\right\}$$ **Ans.**

OR

If $y = 2\cos(\log x)+3\sin(\log x)$, prove that $x^2\frac{d^2y}{dx^2}+x\frac{dy}{dx}+y = 0$.

Solution : Given, $y = 2\cos(\log x)+3\sin(\log x)$

On differentiating both sides w.r.t. x, we get

$$\frac{dy}{dx} = -2\sin(\log x).\frac{1}{x}+3\cos(\log x).\frac{1}{x}$$

$$\Rightarrow x\frac{dy}{dx} = -2\sin(\log x)+3\cos(\log x)$$

Again differentiating both sides w.r.t. x, we get

$$x\frac{d^2y}{dx^2}+\frac{dy}{dx} = -2\cos(\log x).\frac{1}{x}-3\sin(\log x).\frac{1}{x}$$

$$\Rightarrow \quad x^2\frac{d^2y}{dx^2}+x.\frac{dy}{dx} = -2\sin(\log x)+3\cos(\log x)$$

$$\Rightarrow \quad x^2\frac{d^2y}{dx^2}+x\frac{dy}{dx} = -y$$

$$\Rightarrow x^2\frac{d^2y}{dx^2}+x\frac{dy}{dx}+y = 0$$ **Hence Proved.**

11. If $x = a\sin 2t(1+\cos 2t)$ and $y = b\cos 2t(1-\cos 2t)$, find $\frac{dy}{dx}$ at $t = \frac{\pi}{4}$. [4]

Solution : We have, $x = a\sin 2t(1+\cos 2t)$

$$\Rightarrow \frac{dx}{dt} = 2a\cos 2t\,(1+\cos 2t) - 2a\sin^2 2t$$

and $y = b\cos 2t\,(1-\cos 2t)$

$$\Rightarrow \frac{dy}{dt} = 2b\cos 2t\sin 2t - 2b\sin 2t\,(1-\cos 2t)$$

$$\therefore \frac{dy}{dx} = \frac{dy}{dt}\times\frac{dt}{dx}$$

$$= \frac{2b\cos 2t\sin 2t - 2b\sin 2t\,(1-\cos 2t)}{2a\cos 2t(1+\cos 2t) - 2a\sin^2 2t}$$

$$= \frac{b[\cos 2t\sin 2t - \sin 2t(1-\cos 2t)]}{a[\cos 2t(1+\cos 2t) - \sin^2 2t]}$$

Since, we know that at $t = \frac{\pi}{4}$, $\sin 2t = 1$ and $\cos 2t = 0$

So, $$\left(\frac{dy}{dx}\right)_{t=\frac{\pi}{4}} = \frac{b[0.1-1(1-0)]}{a[0(1+0)-1^2]} = \frac{b}{a}$$ **Ans.**

12. The equation of tangent at (2, 3) on the curve $y^2 = ax^3 + b$ is $y = 4x - 5$. Find the value of a and b. **[4]**

Solution : Since the point (2, 3) lies on the curve $y^2 = ax^3 + b$

$$\Rightarrow (3)^2 = a(2)^3 + b$$

$$\Rightarrow 9 = 8a + b \quad \text{...(i)}$$

and $$y^2 = ax^3 + b$$

On differentiating w.r.t. x,

$$2y\frac{dy}{dx} = 3ax^2$$

$$\Rightarrow \frac{dy}{dx} = \frac{3ax^2}{2y}$$

$$\left(\frac{dy}{dx}\right)_{(2,3)} = \frac{3a(2)^2}{2(3)} = 2a$$

The required equation of the tangent at (2, 3) is

$$(y-3) = 2a\,(x-2)$$

$$\Rightarrow y - 3 = 2ax - 4a$$

$$\Rightarrow y = 2ax + 3 - 4a$$

Comparing the equation with $y = 4x - 5$, we get

$$4 = 2a \Rightarrow a = 2$$

From equation (i), we get

$$9 = 8(2) + b \Rightarrow b = -7$$ **Ans.**

13. Find : $\int \frac{x^2}{x^4 + x^2 - 2}dx$ **[4]**

Solution : Let,

$$I = \int \frac{x^2}{x^4+x^2-2}dx = \int \frac{x^2}{(x^2-1)(x^2+2)}dx$$

Let $x^2 = y$

Then, $$\frac{y}{(y-1)(y+2)} = \frac{A}{(y-1)} + \frac{B}{(y+2)} \quad \text{...(i)}$$

$$y = A\,(y+2) + B\,(y-1) \quad \text{...(ii)}$$

Putting $y = 1$ and $y = -2$ successively in (ii), we get

$$A = \frac{1}{3} \text{ and } B = \frac{2}{3}$$

Substituting the values of A and B in (i), we obtain

$$\frac{y}{(y-1)(y+2)} = \frac{1}{3(y-1)} + \frac{2}{3(y+2)}$$

Replacing y by x^2, we obtain

$$\frac{x^2}{(x^2-1)(x^2+2)} = \frac{1}{3(x^2-1)} + \frac{2}{3(x^2+2)}$$

$$\therefore I = \int \frac{x^2}{x^4+x^2-2}dx = \frac{1}{3}\int\frac{1}{(x^2-1)}dx + \frac{2}{3}\int\frac{1}{(x^2+2)}dx$$

$$= \frac{1}{3}\cdot\frac{1}{2}\log\left|\frac{x-1}{x+1}\right| + \frac{2}{3}\cdot\frac{1}{\sqrt{2}}\tan^{-1}\left(\frac{x}{\sqrt{2}}\right) + C$$

$$= \frac{1}{6}\log\left|\frac{x-1}{x+1}\right| + \frac{\sqrt{2}}{3}\tan^{-1}\left(\frac{x}{\sqrt{2}}\right) + C$$ **Ans**

14. Evaluate : $\int_0^{\pi/2} \frac{\sin^2 x}{\sin x + \cos x}dx$ **[4]**

Solution : We have,

$$I = \int_0^{\pi/2} \frac{\sin^2 x}{\sin x + \cos x}dx \quad \text{...(i)}$$

$$\Rightarrow I = \int_0^{\pi/2} \frac{\sin^2\left(\frac{\pi}{2}-x\right)}{\sin\left(\frac{\pi}{2}-x\right) + \cos\left(\frac{\pi}{2}-x\right)}dx$$

$$\left[\text{Using} : \int_0^a f(x)dx = \int_0^a f(a-x)dx\right]$$

$$\Rightarrow I = \int_0^{\pi/2} \frac{\cos^2 x}{\cos x + \sin x}dx \quad \text{...(ii)}$$

Adding equation (i) and (ii), we get

$$2I = \int_0^{\pi/2} \frac{\sin^2 x}{\sin x + \cos x} + \frac{\cos^2 x}{\sin x + \cos x}dx$$

$$= \int_0^{\pi/2} \frac{1}{\sin x + \cos x}dx$$

$$\Rightarrow 2I = \int_0^{\pi/2} \frac{1}{\frac{2\tan x/2}{1+\tan^2 x/2} + \frac{1-\tan^2 x/2}{1+\tan^2 x/2}}dx$$

$$\Rightarrow 2I = \int_0^{\pi/2} \frac{1+\tan^2 x/2}{2\tan x/2 + 1 - \tan^2 x/2}dx$$

$$= \int_0^{\pi/2} \frac{\sec^2 x/2}{2\tan x/2 + 1 - \tan^2 x/2}dx$$

Let $\tan\frac{x}{2}=t$.

Then, $\sec^2\frac{x}{2}\frac{1}{2}dx=dt \Rightarrow \sec^2\frac{x}{2}dx=2dt$

Also, $x=0 \Rightarrow t=\tan 0=0$ and $x=\frac{\pi}{2} \Rightarrow t=\tan\frac{\pi}{4}=1$

$$\therefore \quad 2I = \int_0^1 \frac{2dt}{2t+1-t^2} = 2\int_0^1 \frac{1}{(\sqrt{2})^2-(t-1)^2}dt$$

$$\Rightarrow \quad 2I = 2\times\frac{1}{2\sqrt{2}}\left[\log\left|\frac{\sqrt{2}+t-1}{\sqrt{2}-t+1}\right|\right]_0^1$$

$$\Rightarrow \quad 2I = \frac{1}{\sqrt{2}}\left\{\log\left(\frac{\sqrt{2}}{\sqrt{2}}\right)-\log\left(\frac{\sqrt{2}-1}{\sqrt{2}+1}\right)\right\}$$

$$\Rightarrow \quad 2I = \frac{1}{\sqrt{2}}\left\{0-\log\left(\frac{\sqrt{2}-1}{\sqrt{2}+1}\right)\right\}$$

$$\Rightarrow \quad 2I = \frac{1}{\sqrt{2}}\left\{\log\left(\frac{\sqrt{2}-1}{\sqrt{2}+1}\right)^{-1}\right\}$$

$$\Rightarrow \quad 2I = \frac{1}{\sqrt{2}}\left\{\log\left(\frac{\sqrt{2}+1}{\sqrt{2}-1}\right)\right\}$$

$$\Rightarrow \quad I = \frac{1}{2\sqrt{2}}\log\left(\frac{\sqrt{2}+1}{\sqrt{2}-1}\right)$$ **Ans.**

OR

Evaluate : $\int_0^{3/2}|x\cos\pi x|dx$

Solution : We have, $0<x<\frac{3}{2} \Rightarrow 0<\pi x<\frac{3\pi}{2}$

Now, $\quad 0<x<\frac{1}{2}$

$\qquad 0<\pi x<\frac{\pi}{2}$

$\Rightarrow \quad \cos\pi x>0$

$\Rightarrow x\cos\pi x>0$

$\Rightarrow |x\cos\pi x| = x\cos\pi x$ for $0<\pi x<\frac{\pi}{2}$

and $\quad \frac{1}{2}<x<\frac{3}{2}$

$\Rightarrow \quad \frac{\pi}{2}<\pi x<\frac{3\pi}{2}$

$\Rightarrow \quad \cos\pi x<0$

$\Rightarrow x\cos\pi x<0$

$\Rightarrow |x\cos\pi x| = -x\cos\pi x$ for $\pi<\pi x<\frac{3\pi}{2}$

$\therefore \int_0^{3/2}|x\cos\pi x|dx$

$$= \int_0^{1/2}|x\cos\pi x|dx+\int_{1/2}^{3/2}|x\cos\pi x|dx$$

$$= \int_0^{1/2}(x\cos\pi x)dx+\int_{1/2}^{3/2}(-x\cos\pi x)dx$$

$$= \int_0^{1/2}(x\cos\pi x)dx-\int_{1/2}^{3/2}(x\cos\pi x)dx$$

$$= \left[\left(\left|\frac{x\sin\pi x}{\pi}\right|_0^{1/2}-\int_0^{1/2}\frac{\sin\pi x}{\pi}dx\right)-\left(\left|\frac{x\sin\pi x}{\pi}\right|_{1/2}^{3/2}-\int_{1/2}^{3/2}\frac{\sin\pi x}{\pi}dx\right)\right]$$

$$= \left[\left\{\frac{1}{\pi}\left(\frac{1}{2}\frac{\sin\pi}{2}-0\right)+\left.\frac{\cos\pi x}{\pi^2}\right|_0^{1/2}\right\}-\left\{\frac{1}{\pi}\left(\frac{3}{2}\sin\frac{3\pi}{2}-\frac{1}{2}\sin\frac{\pi}{2}\right)+\left.\frac{\cos\pi x}{\pi^2}\right|_{1/2}^{3/2}\right\}\right]$$

$$=\frac{1}{2\pi}+\frac{1}{\pi^2}\left(\frac{\cos\pi}{2}-\cos 0\right)-\frac{1}{\pi}\left(\frac{3}{2}(-1)-\frac{1}{2}(1)\right)+\frac{1}{\pi^2}\left(\frac{\cos 3\pi}{2}-\frac{\cos\pi}{2}\right)\}$$

$$= \frac{1}{2\pi}+\frac{1}{\pi^2}(0-1)-\frac{1}{\pi}\left(-\frac{3}{2}-\frac{1}{2}\right)+\frac{1}{\pi^2}(0-0)$$

$$= \frac{1}{2\pi}-\frac{1}{\pi^2}+\frac{2}{\pi}$$

$$= \frac{1}{2\pi}+\frac{2}{\pi}-\frac{1}{\pi^2}$$

$$= \frac{\pi+4\pi-2}{2\pi^2}$$

$$= \frac{5\pi-2}{2\pi^2}$$ **Ans.**

15. Find : $\int(3x+1)\sqrt{4-3x-2x^2}\,dx$ **[4]**

Solution : Let $3x+1 = \lambda\frac{d}{dx}(4-3x-2x^2)+\mu$

Then, $\qquad 3x+1 = \lambda(-3-4x)+\mu$

$\Rightarrow \qquad 3x+1 = -4\lambda x+(-3\lambda+\mu)$

Comparing the coefficients of like powers of x, we get

$$-4\lambda = 3 \text{ and } -3\lambda+\mu=1$$

$$\Rightarrow \quad \lambda = \frac{-3}{4} \text{ and } \mu=\frac{-5}{4}$$

Let $\qquad I = \int(3x+1)\sqrt{4-3x-2x^2}\,dx$

$$I=\int\left\{-\frac{3}{4}(-3-4x)-\frac{5}{4}\right\}\sqrt{4-3x-2x^2}\,dx$$

$$= -\frac{3}{4}\int(-3-4x)\sqrt{4-3x-2x^2}\,dx - \frac{5}{4}\int\sqrt{4-3x-2x^2}\,dx$$

$$= -\frac{3}{4}\int\sqrt{t}\,dt-\frac{5}{4}\int\sqrt{-2\left(x^2+\frac{3}{2}x+\frac{9}{16}-\frac{9}{16}-2\right)}\,dx$$

where, $t = 4 - 3x - 2x^2$

$\Rightarrow \quad dt = (-3-4x)\,dx$

$$= -\frac{3}{4}\left(\frac{t^{3/2}}{3/2}\right) + c_1 - \frac{5\sqrt{2}}{4}\int\sqrt{-\left(x+\frac{3}{4}\right)^2 + \left(\frac{\sqrt{41}}{4}\right)^2}\,dx$$

$$= -\frac{1}{2}(4-3x-2x^2)^{3/2} + c_1$$

$$-\frac{5}{2\sqrt{2}}\left\{\frac{1}{2}\left(x+\frac{3}{4}\right)\sqrt{\left(\frac{\sqrt{41}}{4}\right)^2 - \left(x+\frac{3}{4}\right)^2}\right\}$$

$$-\frac{5}{2\sqrt{2}}\times\frac{1}{2}\left(\frac{\sqrt{41}}{4}\right)^2\sin^{-1}\left(\frac{x+\frac{3}{4}}{\sqrt{41}/4}\right) + c_2$$

$$= -\frac{1}{2}(4-3x-2x^2)^{3/2} - \frac{5}{4\sqrt{2}}\left\{\left(x+\frac{3}{4}\right)\sqrt{\frac{4-3x-2x^2}{2}}\right\}$$

$$-\frac{5}{4\sqrt{2}}\times\frac{41}{16}\sin^{-1}\left(\frac{4x+3}{\sqrt{41}}\right) + c$$

where $(c = c_1 + c_2)$

$$= -\frac{1}{2}(4-3x-2x^2)^{3/2} - \frac{5}{8}\left(x+\frac{3}{4}\right)\sqrt{4-3x-2x^2}$$

$$+\frac{5\times 41}{4\sqrt{2}\times 16}\sin^{-1}\left(\frac{4x+3}{\sqrt{41}}\right) + c$$

$$= -\frac{1}{2}\left(4-3x-2x^2\right)^{3/2} - \frac{5}{8}\left(\frac{4x+3}{4}\right)\sqrt{4-3x-2x^2}$$

$$+\frac{5\times 41\sqrt{2}}{8\times 16}\sin^{-1}\left(\frac{4x+3}{\sqrt{41}}\right) + c$$

$$= -\frac{1}{2}\left(4-3x-2x^2\right)^{3/2} - \frac{5}{8}\left[\left(\frac{4x+3}{4}\right)\sqrt{4-3x-2x^2}\right.$$

$$\left. + \frac{41\sqrt{2}}{16}\sin^{-1}\left(\frac{4x+3}{\sqrt{41}}\right) + c\right] \quad \textbf{Ans.}$$

16. Solve the differential equation :

$$y + x\frac{dy}{dx} = x - y\frac{dy}{dx}$$ **[4]**

Solution : We have, $y + x\frac{dy}{dx} = x - y\frac{dy}{dx}$

$$\Rightarrow \quad x\frac{dy}{dx} + y\frac{dy}{dx} = x - y$$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{x-y}{x+y} \quad \text{...(i)}$$

Which is a homogeneous differential equation.

Putting $y = Vx \Rightarrow \frac{dy}{dx} = V + x\frac{dV}{dx}$ in (i), we get

$$V + x\frac{dV}{dx} = \frac{x - Vx}{x + Vx}$$

$$\Rightarrow \quad V + x\frac{dV}{dx} = \frac{1-V}{1+V}$$

$$\Rightarrow \quad x\frac{dV}{dx} = \frac{1-V}{1+V} - V$$

$$\Rightarrow \quad x\frac{dV}{dx} = \frac{1-V-V-V^2}{1+V}$$

$$\Rightarrow \quad x\frac{dV}{dx} = \frac{1-2V-V^2}{1+V}$$

$$\Rightarrow \quad \frac{1+V}{1-2V-V^2}dV = \frac{dx}{x}, \quad x \neq 0$$

$$\Rightarrow \quad \frac{1+V}{V^2+2V-1}dV = \frac{-dx}{x} \quad \text{...(ii)}$$

Putting $t = V^2 + 2V - 1$

$\Rightarrow \quad dt = (2V+2)\,dV$

$\Rightarrow \quad \frac{1}{2}dt = (V+1)\,dV$

Now, equation (ii) becomes

$$\frac{1}{2t}dt = \frac{-dx}{x}$$

On integrating above equation we get,

$$\frac{1}{2}\int\frac{1}{t}dt = -\int\frac{dx}{x}$$

$$\Rightarrow \quad \frac{1}{2}\log|t| = -\log|x| + \log C$$

$$\Rightarrow \frac{1}{2}\log|V^2+2V-1|^{1/2} = \log\left|\frac{C}{x}\right|$$

$$\Rightarrow \quad V^2 + 2V - 1 = \left(\frac{C}{x}\right)^2$$

$$\Rightarrow \quad \left(\frac{y}{x}\right)^2 + 2\left(\frac{2y}{x}\right) - 1 = \frac{C^2}{x^2} \quad \left(\because V = \frac{y}{x}\right)$$

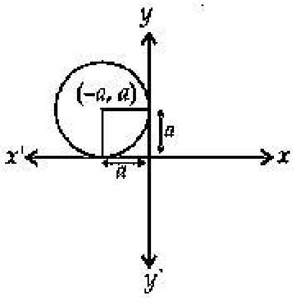
$$\Rightarrow \quad \frac{y^2}{x^2} + \frac{2y}{x} - 1 = \frac{C^2}{x^2}$$

$$\Rightarrow \quad y^2 + 2xy - x^2 = C^2 \quad \textbf{Ans.}$$

17. Form the differential equation of the family of circles in the second quadrant and touching the coordinate axes. [4]

Solution : The equation of circles in the second quadrant which touch the coordinate axes is

$$(x+a)^2 + (y-a)^2 = a^2, \; a \in R \quad \text{...(i)}$$



where a is a parameter. This equation contains one arbitrary constant. So we shall differentiate it once only and we shall get a differential equation of first order.

Differentiating (i) w.r.t. x, we get

$$2(x+a)+2(y-a)\frac{dy}{dx} = 0$$

$$\Rightarrow \quad x+a+(y-a)\frac{dy}{dx} = 0$$

$$\Rightarrow \quad a = -\left(\frac{x+y\frac{dy}{dx}}{1-\frac{dy}{dx}}\right)$$

$$\Rightarrow \quad a = \frac{x+Py}{P-1}, \text{ where } P = \frac{dy}{dx}$$

Substituting the value of a in (i), we get

$$\left(x+\frac{x+Py}{P-1}\right)^2+\left(y-\frac{x+Py}{P-1}\right)^2 = \left(\frac{x+Py}{P-1}\right)^2$$

$$\Rightarrow (xP-x+x+yP)^2+(yP-y-x-yP)^2 = (x+yP)^2$$

$$\Rightarrow \quad (x+y)^2P^2+(x+y)^2 = (x+yP)^2$$

$$\Rightarrow \quad (x+y)^2(P^2+1) = (x+yP)^2$$

$$\Rightarrow \quad (x+y)^2\left[\left(\frac{dy}{dx}\right)^2+1\right] = \left(x+y\frac{dy}{dx}\right)^2$$

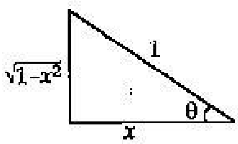
This is the required differential equation representing the given family of circles. **Ans.**

18. Solve the equation for x : $\sin^{-1}x+\sin^{-1}(1-x) = \cos^{-1}x$. **[4]**

Solution : We have, $\sin^{-1}x+\sin^{-1}(1-x) = \cos^{-1}x$

$$\Rightarrow \sin^{-1}(x\sqrt{1-(1-x)^2}+(1-x)\sqrt{1-x^2}) = \cos^{-1}x$$

$$\Rightarrow \sin^{-1}(x\sqrt{1-(1+x^2-2x)}+(1-x)\sqrt{1-x^2}) = \cos^{-1}x$$



Let $\quad \cos^{-1}x = \theta$

$\Rightarrow \quad x = \cos\theta$

and $\quad \sin\theta = \sqrt{1-x^2}$

$$\Rightarrow \quad \theta = \sin^{-1}(\sqrt{1-x^2}) = \cos^{-1}x$$

$$\Rightarrow \sin^{-1}(x\sqrt{2x-x^2}+(1-x)\sqrt{1-x^2}) = \sin^{-1}(\sqrt{1-x^2})$$

$$\Rightarrow \quad x\sqrt{2x-x^2}+(1-x)\sqrt{1-x^2} = \sqrt{1-x^2}$$

$$\Rightarrow \quad x\sqrt{2x-x^2} = \sqrt{1-x^2}.(1-1+x)$$

$$\Rightarrow \quad x\left[\sqrt{2x-x^2}-\sqrt{1-x^2}\right] = 0$$

$$\Rightarrow \quad x = 0 \text{ or } \sqrt{2x-x^2} = \sqrt{1-x^2}$$

$$\Rightarrow \quad x = 0 \text{ or } 2x-x^2 = 1-x^2$$

$$\Rightarrow \quad x = 0 \text{ or } x = \frac{1}{2}$$ **Ans.**

OR

If $\cos^{-1}\frac{x}{a}+\cos^{-1}\frac{y}{b} = \alpha$, prove that :

$$\frac{x^2}{a^2}-2\frac{xy}{ab}\cos\alpha+\frac{y^2}{b^2} = \sin^2\alpha$$

Solution : We have, $\cos^{-1}\frac{x}{a}+\cos^{-1}\frac{y}{b} = \alpha$

$$\Rightarrow \quad \cos^{-1}\left(\frac{x}{a}\frac{y}{b}+\sqrt{1-\frac{x^2}{a^2}}\sqrt{1-\frac{y^2}{b^2}}\right) = \alpha$$

$$\Rightarrow \quad \frac{xy}{ab}+\sqrt{1-\frac{x^2}{a^2}}\sqrt{1-\frac{y^2}{b^2}} = \cos\alpha$$

$$\Rightarrow \quad \sqrt{1-\frac{x^2}{a^2}}\sqrt{1-\frac{y^2}{b^2}} = \cos\alpha-\frac{xy}{ab}$$

On squaring both sides, we get

$$\left(1-\frac{x^2}{a^2}\right)\left(1-\frac{y^2}{b^2}\right) = \left(\cos\alpha-\frac{xy}{ab}\right)^2$$

$$\Rightarrow 1-\frac{x^2}{a^2}-\frac{y^2}{b^2}+\frac{x^2y^2}{a^2b^2} = \cos^2\alpha+\frac{x^2y^2}{a^2b^2}-\frac{2xy}{ab}\cos\alpha$$

$$\Rightarrow \quad 1-\frac{x^2}{a^2}-\frac{y^2}{b^2} = 1-\sin^2\alpha-\frac{2xy}{ab}\cos\alpha$$

$$\Rightarrow \quad \frac{x^2}{a^2}-\frac{2xy}{ab}\cos\alpha+\frac{y^2}{b^2} = \sin^2\alpha$$ **Hence Proved.**

19. A trust invested some money in two types of bond. The first bond pays 10% interest and second bond pays 12% interest. The trust received ₹ 2,800 as interest. However, if trust had interchanged money in bonds, they would have got ₹ 100 less as interest. Using matrix method, find the amount invested by the trust. Interest received on this amount will be given to Helpage India as donation. Which value is reflected in this question ? **[4]**

Solution : Let the amount invested by the trust in first and second bond be x and y respectively.

$$\text{Interest from first bond} = \frac{10\times x\times 1}{100} = \frac{10x}{100}$$

Interest from second bond = $\frac{12 \times y \times 1}{100} = \frac{12y}{100}$

Interest received by trust = ₹ 2,800

According to the question,

$$\frac{10x}{100} + \frac{12y}{100} = 2{,}800$$

$$\Rightarrow \quad 10x + 12y = 2{,}80{,}000 \quad \text{...(i)}$$

and

$$\frac{12x}{100} + \frac{10y}{100} = 2{,}700$$

$$\Rightarrow \quad 12x + 10y = 2{,}70{,}000 \quad \text{...(ii)}$$

This system of equations can be written in matrix form as follows :

$$\begin{bmatrix} 10 & 12 \\ 12 & 10 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 2{,}80{,}000 \\ 2{,}70{,}000 \end{bmatrix}$$

or $\quad AX = B,$

where $A = \begin{bmatrix} 10 & 12 \\ 12 & 10 \end{bmatrix}$, $X = \begin{bmatrix} x \\ y \end{bmatrix}$ and $B = \begin{bmatrix} 2{,}80{,}000 \\ 2{,}70{,}000 \end{bmatrix}$

Now, $|A| = \begin{vmatrix} 10 & 12 \\ 12 & 10 \end{vmatrix} = 100 - 144 = -44 \neq 0$

So, A^{-1} exists and the solution of the given system of equations is given by

$$X = A^{-1}B$$

Let c_{ij} be the cofactor of a_{ij} in $A = [a_{ij}]$. Then,

$c_{11} = 10, c_{12} = -12, c_{21} = -12, c_{22} = 10$

$$\therefore \text{adj } A = \begin{bmatrix} 10 & -12 \\ -12 & 10 \end{bmatrix}^T = \begin{bmatrix} 10 & -12 \\ -12 & 10 \end{bmatrix}$$

$$\text{So, } A^{-1} = \frac{1}{|A|}(\text{adj } A) = -\frac{1}{44}\begin{bmatrix} 10 & -12 \\ -12 & 10 \end{bmatrix}$$

Hence, the solution is given by

$$X = A^{-1}B = -\frac{1}{44}\begin{bmatrix} 10 & -12 \\ -12 & 10 \end{bmatrix}\begin{bmatrix} 2{,}80{,}000 \\ 2{,}70{,}000 \end{bmatrix}$$

$$\begin{bmatrix} x \\ y \end{bmatrix} = -\frac{1}{44}\begin{bmatrix} 28{,}00{,}000 - 32{,}40{,}000 \\ -33{,}60{,}000 + 27{,}00{,}000 \end{bmatrix}$$

$$\begin{bmatrix} x \\ y \end{bmatrix} = -\frac{1}{44}\begin{bmatrix} -4{,}40{,}000 \\ -6{,}60{,}000 \end{bmatrix} = \begin{bmatrix} 10{,}000 \\ 15{,}000 \end{bmatrix}$$

$\Rightarrow x = 10{,}000$ and $y = 15{,}000$

$\Rightarrow A = x + y = 10{,}000 + 15{,}000 =$ ₹ 25,000

Hence, the amount invested by the trust is ₹ 25,000.

Value : Giving help to those in need is a humanitarian act. **Ans.**

SECTION — C

20. There are two types of fertilisers 'A' and 'B'. 'A' consists of 12% nitrogen and 5% phosphoric acid whereas 'B' consists of 4% nitrogen and 5% phosphoric acid. After testing the soil conditions, farmer finds that he needs at least 12 kg of nitrogen and 12 kg of phosphoric acid for his crops. If 'A' costs ₹ 10 per kg and 'B' cost ₹ 8 per kg, then graphically determine how much of each type of fertiliser should be used so that nutrient requirements are met at a minimum cost. [6]

Solution : Let the quantity of fertiliser A and B be x and y respectively.

To minimize : Z = ₹ $(10x + 8y)$

Subject to the constraints :

$$\frac{12}{100}x + \frac{4}{100}y \geq 12$$

or $\quad 12x + 4y \geq 1200$

and $\quad \frac{5x}{100} + \frac{5y}{100} \geq 12$

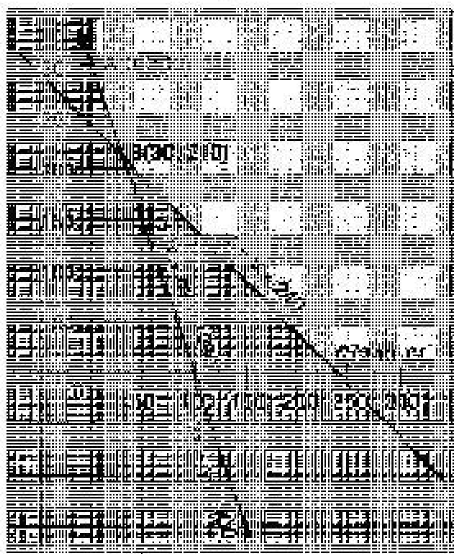
or $\quad 5x + 5y \geq 1200$

and $\quad x \geq 0, y \geq 0$

i.e., $\quad 3x + y \geq 300,$

$x + y \geq 240,$

$x \geq 0, y \geq 0$



Corner Points	$Z = 10x + 8y$
A (0,300)	$Z = 10 \times 0 + 8 \times 300 =$ ₹ 2400
B (30,210)	$Z = 10 \times 30 + 8 \times 210 =$ ₹ 1980
C (240,0)	$Z = 10 \times 240 + 8 \times 0 =$ ₹ 2400

The region of $10x + 8y < 1980$ has no point in common to the feasible region.

So, Z is minimum for $x = 30$ and $y = 210$ and the minimum value of Z is ₹ 1980.

Hence, the quantity of fertilizer A is 30 kg and of fertilizer B is 210 kg. **Ans.**

21. Five bad oranges are accidently mixed with 20 good ones. If four oranges are drawn one by one successively with replacement, then find the probability distribution of number of bad oranges drawn. Hence find the mean and variance of the distribution. [6]

Solution : Let X denotes the number of bad oranges in a draw of 4 oranges from a group of 20 good oranges and 5 bad oranges. Since there are 5 bad oranges in the group, therefore X can take values, 0, 1, 2, 3, 4.

Now, P(X = 0) = Probability of getting no bad orange.

P(X = 0) = Probability of getting 4 good oranges

$$= \left(\frac{20}{25}\right)^4 \cdot {}^4C_0$$

P(X = 1) = Probability of getting one bad orange

$$= \frac{5}{25} \times \left(\frac{20}{25}\right)^3 \cdot {}^4C_1$$

P(X = 2) = Probability of getting two bad oranges

$$= \left(\frac{5}{25}\right)^2 \times \left(\frac{20}{25}\right)^2 \cdot {}^4C_2$$

P(X = 3) = Probability of getting three bad oranges

$$= \left(\frac{5}{25}\right)^3 \times \frac{20}{25} \cdot {}^4C_3$$

P(X = 4) = Probability of getting four bad oranges

$$= \left(\frac{5}{25}\right)^4 \cdot {}^4C_4$$

Computation of Mean and Variance

x_i	$p_i = p(X = x_i)$	$p_i x_i$	$p_i x_i^2$
0	$\frac{256}{625}$	0	0
1	$\frac{256}{625}$	$\frac{256}{625}$	$\frac{256}{625}$
2	$\frac{96}{625}$	$\frac{192}{625}$	$\frac{384}{625}$
3	$\frac{16}{625}$	$\frac{48}{625}$	$\frac{144}{625}$
4	$\frac{1}{625}$	$\frac{4}{625}$	$\frac{16}{625}$
		$\Sigma p_i x_i = \frac{500}{625}$	$\Sigma p_i x_i^2 = \frac{800}{625}$

We have, $\Sigma p_i x_i = \frac{500}{625} = \frac{4}{5}$ and $\Sigma p_i x_i^2 = \frac{800}{625} = \frac{32}{25}$

$\therefore$ $\overline{X}$ = Mean = $\Sigma p_i x_i = \frac{4}{5}$

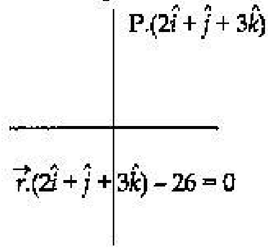
and Var(X) = $\Sigma p_i x_i^2 - (\Sigma p_i x_i)^2 = \frac{32}{25} - \frac{16}{25} = \frac{16}{25}$

Hence, mean = $\frac{4}{5}$ and variance = $\frac{16}{25}$ **Ans.**

22. Find the position vector of the foot of perpendicular and the perpendicular distance from the point P with position vector $2\hat{i} + 3\hat{j} + 4\hat{k}$ to the plane $\vec{r}\cdot(2\hat{i} + \hat{j} + 3\hat{k}) - 26 = 0$. Also find image of P in the plane. [6]

Solution : Let L be the foot of the perpendicular drawn from $P(2\hat{i} + 3\hat{j} + 4\hat{k})$ on the plane $\vec{r}\cdot(2\hat{i} + \hat{j} + 3\hat{k}) - 26 = 0$

We know that the position vector of the foot of perpendicular from the point P with position vector $\vec{a}$ from the plane $\vec{r}\cdot\vec{n} = d$ is given by



$$\vec{PL} = \frac{[d - (\vec{a}\cdot\vec{n})]\vec{n}}{|\vec{n}|^2}$$

$$= \frac{[26 - (2\hat{i} + 3\hat{j} + 4\hat{k})(2\hat{i} + \hat{j} + 3\hat{k})](2\hat{i} + \hat{j} + 3\hat{k})]}{(\sqrt{4+1+9})^2}$$

$$= \frac{[26 - (4+3+12)](2\hat{i} + \hat{j} + 3\hat{k})}{14} = \frac{7}{14}(2\hat{i} + \hat{j} + 3\hat{k})$$

$$= \hat{i} + \frac{1}{2}\hat{j} + \frac{3}{2}\hat{k}$$

and the perpendicular distance is given by $\frac{|\vec{a}\cdot\vec{n} - d|}{|\vec{n}|}$

So, required distance

$$= \frac{|(2\hat{i} + 3\hat{j} + 4\hat{k})(2\hat{i} + \hat{j} + 3\hat{k}) - 26|}{\sqrt{14}}$$

$$= \frac{7}{\sqrt{14}}$$

Let Q be the image of the point $P(2\hat{i} + 3\hat{j} + 4\hat{k})$ to the plane $\vec{r}\cdot(2\hat{i} + \hat{j} + 3\hat{k}) - 26 = 0$. Then PQ is normal to the plane.

Therefore, equation of line PQ is

$$\vec{r} = (2\hat{i} + 3\hat{j} + 4\hat{k}) + \lambda(2\hat{i} + \hat{j} + 3\hat{k})$$

Since Q lies on line PQ. So, let the position vector of Q be $(2\hat{i}+3\hat{j}+4\hat{k})+\lambda(2\hat{i}+\hat{j}+3\hat{k})$

$$=(2+2\lambda)\hat{i}+(3+\lambda)\hat{j}+(4+3\lambda)\hat{k}$$

Since, R is the mid-point of PQ. Therefore, position vector of R is

$$\frac{[(2+2\lambda)\hat{i}+(3+\lambda)\hat{j}+(4+3\lambda)\hat{k}]+(2\hat{i}+3\hat{j}+4\hat{k})}{2}$$

$$=(2+\lambda)\hat{i}+(3+\lambda/2)\hat{j}+(4+3\lambda/2)\hat{k}$$

Since R lies on the plane $\vec{r}.[2\hat{i}+\hat{j}+3\hat{k}]-26=0$

$$\therefore \left\{(2+\lambda)\hat{i}+(3+\lambda/2)\hat{j}+(4+3\lambda/2)\hat{k}\right\}.$$

$$(2\hat{i}+\hat{j}+3\hat{k})-26=0$$

$$\Rightarrow \quad 4+2\lambda+3+\frac{\lambda}{2}+12+\frac{9\lambda}{2}-26=0$$

$$\Rightarrow \quad \lambda = \frac{7}{7}=1$$

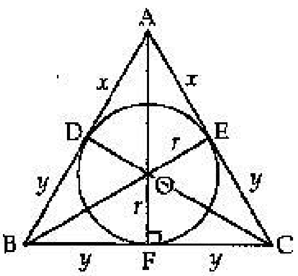
Thus, the position vector of Q is

$$4\hat{i}+4\hat{j}+7\hat{k}$$ **Ans.**

23. Show that the binary operation * on A = R − {− 1} defined as $a * b = a + b + ab$ for all $a, b \in$ A is commutative and associative on A. Also find the identity element of * in A and prove that every element of A is invertible. [6]**

24. Prove that the least perimeter of an isosceles triangle in which a circle of radius r can be inscribed is $6\sqrt{3}r$. [6]

Solution : Let ABC be an isosceles triangle with AB = AC and a circle with centre O and radius r, touching sides AB, BC, CA at D, F, E respectively.



In ΔABC

Let AD = AE = x, BD = BF = y and CF = CE = y

(∵ Tangents drawn from an external point are equal)

Now, ar (ΔABC) = ar (ΔAOB) + ar (ΔAOC) + ar (ΔBOC)

$$\Rightarrow \frac{1}{2}\times 2y\left(r+\sqrt{r^2+x^2}\right) = \frac{1}{2}\{2yr+(x+y)r+(x+y)r\}$$

$$\Rightarrow \quad y\left(r+\sqrt{r^2+x^2}\right) = \frac{1}{2}\{2yr+2(x+y)r\}$$

$$\Rightarrow \quad y\left(r+\sqrt{r^2+x^2}\right) = r(y+x+y)$$

$$\Rightarrow \quad yr+y\left(\sqrt{r^2+x^2}\right) = 2yr+rx$$

$$\Rightarrow \quad \left(\sqrt{r^2+x^2}\right)y = rx+yr$$

Squaring both sides, we get

$$y^2(r^2+x^2) = r^2x^2+y^2r^2+2r^2xy$$

$$\Rightarrow \quad y^2r^2+y^2x^2 = r^2x^2+y^2r^2+2r^2xy$$

$$\Rightarrow \quad y^2x = r^2x+2r^2y$$

$$\Rightarrow \quad x = \frac{2r^2y}{y^2-r^2}$$

Now, P (Perimeter of ΔABC) = $2x + 4y$

$$\Rightarrow \quad P = \frac{4r^2y}{y^2-r^2}+4y$$

Differentiate above equation w.r.t. y, we get

$$\frac{dP}{dy} = \frac{4r^2(y^2-r^2)-4r^2y(2y)}{(y^2-r^2)^2}-4$$

$$= \frac{4r^2\left[y^2-r^2-2y^2\right]}{(y^2-r^2)^2}+4$$

$$\Rightarrow \quad \frac{dP}{dy} = \frac{-4r^2(r^2+y^2)}{(y^2-r^2)^2}+4$$

For maxima and minima of P, put

$$\Rightarrow \quad \frac{dP}{dy} = 0$$

$$\Rightarrow \quad = \frac{-4r^2(r^2+y^2)}{(y^2-r^2)^2}+4=0$$

$$\Rightarrow \quad r^2(r^2+y^2) = (y^2-r^2)^2$$

$$\Rightarrow \quad r^4+r^2y^2 = y^4+r^4-2y^2r^2$$

$$\Rightarrow \quad 3y^2r^2 = y^4$$

$$\Rightarrow \quad y^2 = 3r^2$$

$$\Rightarrow \quad y = \sqrt{3}r$$

Now, again differentiate w.r. to y, we get $\frac{d^2P}{dy^2}$

$$= \frac{-4r^2(2y)(y^2-r^2)^2+4r^2(r^2+y^2)2(y^2-r^2)\times 2y}{(y^2-r^2)^4}$$

**Answer is not given due to the change in present syllabus

$$= \frac{4r^2(y^2-r^2)+[-2y(y^2-r^2)+4y(r^2+y^2)]}{(y^2-r^2)^4}$$

$$= \frac{4r^2y(y^2-r^2)+y[-2y^2+2r^2+4r^2+4y^2]}{(y^2-r^2)^4}$$

$$\Rightarrow \frac{d^2P}{dy^2} = \frac{4r^2y[2y^2+6r^2]}{(y^2-r^2)^3}$$

$$\left.\frac{d^2P}{dy^2}\right|_{y=\sqrt{3}r} = \frac{6\sqrt{3}}{r} => 0$$

Hence, perimeter P of ΔABC is least for $y = \sqrt{3}\, r$

and least perimeter is $P = 4y + \dfrac{4r^2y}{y^2-r^2}$

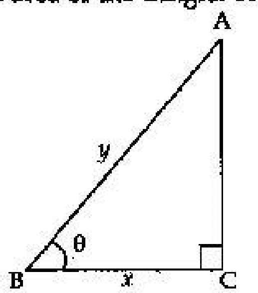
$$= 4\sqrt{3} - \frac{4r^2\sqrt{3}r}{2r^2}$$

$$= 6\sqrt{3}r$$ **Hence Proved.**

OR

If the sum of lengths of hypotenuse and a side of a right angled triangle is given, show that area of triangle is maximum, when the angle between them is $\frac{\pi}{3}$.

Solution: Let ABC be a right angled triangle with base BC = x, AB = y such that $x + y = k$ (constant).
Let θ be the angle between base and hypotenuse.
Let A be the area of the tringle. Then,



$$A = \frac{1}{2} \times BC \times AC$$

$$= \frac{1}{2} x\sqrt{y^2-x^2}$$

$$\Rightarrow A^2 = \frac{x^2}{4}(y^2-x^2) \quad (\because y = k-x)$$

$$\Rightarrow A^2 = \frac{x^2}{4}[(k-x)^2-x^2]$$

$$\Rightarrow A^2 = \frac{k^2x^2-2kx^2}{4} \quad ...(i)$$

Differentiating w.r.t. x, we get

$$2A\frac{dA}{dx} = \frac{2k^2x-6kx^2}{4}$$

$$\Rightarrow \frac{dA}{dx} = \frac{k^2x-3kx^2}{4A}$$

For maximum or minimum, we have

$$\frac{dA}{dx} = 0 \Rightarrow \frac{k^2x-3kx^2}{4A} = 0 \Rightarrow x = \frac{k}{3}$$

Again differentiating (ii) w.r.t. x, we get

$$2\left(\frac{dA}{dx}\right)^2 + 2A\frac{d^2A}{dx^2} = \frac{2k^2-12kx}{4} \quad ....(iii)$$

Putting $\frac{dA}{dx} = 0$ and $x = \frac{k}{3}$ in (iii), we get

$$\frac{d^2A}{dx^2} = \frac{-k^2}{4A} < 0$$

Thus, A is maximum when $= \frac{k}{3}$

Now, and $x = \frac{k}{3}$ and $y = k - x = k - \frac{k}{3} = \frac{2k}{3}$

$$\therefore \quad \cos\theta = \frac{x}{y} = \frac{k/3}{2k/3} = \frac{1}{2}$$

$$\theta = \cos^{-1}\left(\frac{1}{2}\right) = \frac{\pi}{3}$$ **Hence Proved.**

25. Prove that the curves $y^2 = 4x$ and $x^2 = 4y$ divide the area of square bounded by $x = 0$, $x = 4$, $y = 4$ and $y = 0$ into three equal parts. **[6]**

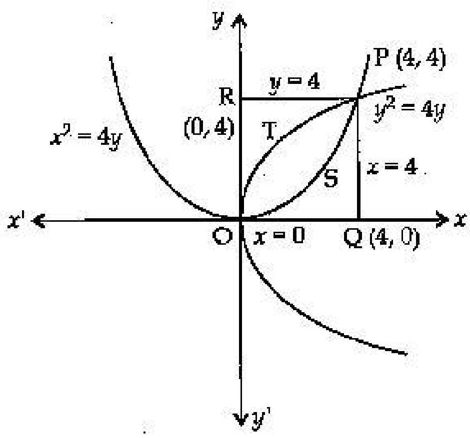
Solution : Let A_1, A_2 and A_3 denote areas OSPQO, OSPTO and OTPRO respectively.

To prove : $A_1 = A_2 = A_3$.

Now, $$A_1 = \int_0^4 \frac{x^2}{4}dx$$

$$= \frac{1}{4}\int_0^4 x^2 dx$$

$$= \frac{1}{4}\left[\frac{x^3}{3}\right]_0^4 = \frac{1}{4} \times \frac{64}{3} = \frac{16}{3} \text{ sq. units}$$



A_2 = (Area bounded by $y^2 = 4x$) – (Area bounded by $x^2 = 4y$)

$$A_2 = \int_0^4 \left(\sqrt{4x} - \frac{x^2}{4}\right) dx$$

$$= \int_0^4 \left(2\sqrt{x} - \frac{x^2}{4}\right) dx$$

$$= \left[\frac{4}{3}x^{3/2} - \frac{x^3}{12}\right]_0^4$$

$$= \left(\frac{4}{3} \times 8 - \frac{64}{12}\right) = \frac{16}{3} \text{ sq. units}$$

and
$$A_3 = \int_0^4 \frac{y^2}{4} dy$$

$$= \frac{1}{4}\left[\frac{y^3}{3}\right]_0^4 = \frac{1}{4} \times \frac{64}{3} = \frac{16}{3} \text{ sq. units.}$$

Hence, $A_1 = A_2 = A_3$. **Hence Proved.**

26. Using properties of determinants, show that ΔABC is isosceles if : [6]

$$\begin{vmatrix} 1 & 1 & 1 \\ 1+\cos A & 1+\cos B & 1+\cos C \\ \cos^2 A + \cos A & \cos^2 B + \cos B & \cos^2 C + \cos C \end{vmatrix} = 0$$

Solution We have,

$$\begin{vmatrix} 1 & 1 & 1 \\ 1+\cos A & 1+\cos B & 1+\cos C \\ \cos^2 A + \cos A & \cos^2 B + \cos B & \cos^2 C + \cos C \end{vmatrix} = 0$$

Applying $C_2 \to C_2 - C_1$ and $C_3 - C_3 \to C_1$

$$\begin{vmatrix} 1 & 0 & 0 \\ 1+\cos A & \cos B - \cos A & \cos C - \cos A \\ \cos^2 A + \cos A & (\cos B - \cos A)(1 + \cos A + \cos B) & (\cos C - \cos A)(1 + \cos A + \cos C) \end{vmatrix} = 0$$

Taking (cos B – cos A) and (cos C – cos A) as common from C_2 and C_3 respectively.

⇒ (cos B – cos A) (cos C – cos A)

$$\begin{vmatrix} 1 & 0 & 0 \\ 1+\cos A & 1 & 1 \\ \cos^2 A + \cos A & 1 + \cos A + \cos B & 1 + \cos A + \cos C \end{vmatrix} = 0$$

⇒ Expanding along R_1

(cos B – cos A) (cos C – cos A)

[1 + cos A + cos C – 1 – cos A – cos B] = 0

⇒ (cos B – cos A) (cos C – cos A) (cos C – cos B) = 0

Either cos B = cos A or cos C = cos A or cos C = cos B

i.e., either BC = AC or BC = AB or AC = AB

Hence, Δ ABC is isosceles. **Hence Proved.**

OR

A shopkeeper has 3 varieties of pens 'A', 'B' and 'C'. Meenu purchased 1 pen of each variety for a total of ₹ 21. Jeevan purchased 4 pens of 'A' variety, 3 pens of 'B' variety and 2 pens of 'C' variety for ₹ 60. While Shikha purchased 6 pens of 'A' variety, 2 pens of 'B' variety and 3 pens of 'C' variety for ₹ 70. Using matrix method, find cost of each variety of pen.

Solution : Let the cost of each variety of pen be ₹ x, ₹ y and ₹ z respectively. Then,

$$x + y + z = 21$$
$$4x + 3y + 2z = 60$$
and
$$6x + 2y + 3z = 70$$

This system of equations can be written in matrix form as follows :

$$\begin{bmatrix} 1 & 1 & 1 \\ 4 & 3 & 2 \\ 6 & 2 & 3 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 21 \\ 60 \\ 70 \end{bmatrix}$$

or $AX = B$

where $A = \begin{bmatrix} 1 & 1 & 1 \\ 4 & 3 & 2 \\ 6 & 2 & 3 \end{bmatrix}$, $X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ and $B = \begin{bmatrix} 21 \\ 60 \\ 70 \end{bmatrix}$

Now, $|A| = \begin{vmatrix} 1 & 1 & 1 \\ 4 & 3 & 2 \\ 6 & 2 & 3 \end{vmatrix}$

$$= 1(9 - 4) - 1(12 - 12) + 1(8 - 18)$$
$$= -5 \neq 0$$

So, A^{-1} exists and the solution of the given system of equation is given by

$$X = A^{-1}B$$

Let c_{ij} be the cofactor of a_{ij} in $A = [a_{ij}]$. Then,

$c_{11} = 5, c_{12} = 0, c_{13} = -10, c_{21} = -1, c_{22} = -3, c_{23} = 4, c_{31} = -1, c_{32} = 2, c_{33} = -1$

$$\therefore \text{adj } A = \begin{bmatrix} 5 & 0 & -10 \\ -1 & -3 & 4 \\ -1 & 2 & -1 \end{bmatrix}^T = \begin{bmatrix} 5 & -1 & -1 \\ 0 & -3 & 2 \\ -10 & 4 & -1 \end{bmatrix}$$

So, $A^{-1} = \frac{1}{|A|}(\text{adj } A) = -\frac{1}{5}\begin{bmatrix} 5 & -1 & -1 \\ 0 & -3 & 2 \\ -10 & 4 & -1 \end{bmatrix}$

Hence, the solution is given by

$$X = A^{-1}B = -\frac{1}{5}\begin{bmatrix} 5 & -1 & -1 \\ 0 & -3 & 2 \\ -10 & 4 & -1 \end{bmatrix}\begin{bmatrix} 21 \\ 60 \\ 70 \end{bmatrix}$$

$$= -\frac{1}{5}\begin{bmatrix} 105-60-70 \\ 0-180+140 \\ -210+240-70 \end{bmatrix}$$

$$= -\frac{1}{5}\begin{bmatrix} -25 \\ -40 \\ -40 \end{bmatrix} = \begin{bmatrix} 5 \\ 8 \\ 8 \end{bmatrix}$$

Hence, the cost of each variety of pen are ₹ 5, ₹ 8 and ₹ 8 respectively. **Ans.**

●●

All questions are same in Outside Delhi Set II and Set III

Mathematics 2016 (Delhi) SET I

Time allowed : 3 hours **Maximum marks : 100**

SECTION — A

1. Find the maximum value of $\begin{vmatrix} 1 & 1 & 1 \\ 1 & 1+\sin\theta & 1 \\ 1 & 1 & 1+\cos\theta \end{vmatrix}$. [1]

Soution : Let $\Delta = \begin{vmatrix} 1 & 1 & 1 \\ 1 & 1+\sin\theta & 1 \\ 1 & 1 & 1+\cos\theta \end{vmatrix}$

On applying $R_1 \to R_1 - R_2$ and $R_2 \to R_2 - R_3$, we get

$$\Delta = \begin{vmatrix} 0 & -\sin\theta & 0 \\ 0 & \sin\theta & -\cos\theta \\ 1 & 1 & 1+\cos\theta \end{vmatrix}$$

On expanding along R_1, we get

$$\Delta = 0 + \sin\theta\,(0 + \cos\theta) + 0$$
$$= \sin\cos\theta$$
$$= \frac{1}{2}\sin 2\theta$$

Now as $-1 \le \sin 2\theta \le 1$ for all $\theta \in R$

$\Rightarrow -\frac{1}{2} \le \frac{1}{2}\sin 2\theta \le \frac{1}{2}$ for all $\theta \in R$

Clearly, maximum value of Δ is $\frac{1}{2}$. **Ans.**

2. If A is a square matrix such that $A^2 = I$, then find the simplified value of $(A - I)^3 + (A + I)^3 - 7A$. [1]

Solution : Given,

$$(A - I)^3 + (A + I)^3 - 7A$$
$$= A^3 - I^3 - 3A^2I + 3AI^2 + A^3 + I^3 + 3A^2I + 3AI^2 - 7A$$
$$= 2A^3 + 6AI^2 - 7A$$
$$= 2A.A^2 + 6AI^2 - 7A$$
$$= 2AI + 6AI - 7A$$
$$= 8A - 7A$$
$$= A$$ **Ans.**

3. Matrix $A = \begin{bmatrix} 0 & 2b & -2 \\ 3 & 1 & 3 \\ 3a & 3 & -1 \end{bmatrix}$ is given to be symmetric, find values of a and b. [1]

Solution : We have, $A = \begin{bmatrix} 0 & 2b & -2 \\ 3 & 1 & 3 \\ 3a & 3 & -1 \end{bmatrix}$

It is given that the matrix is symmetric.

$\therefore \quad A = A'$

$$\Rightarrow \begin{bmatrix} 0 & 2b & -2 \\ 3 & 1 & 3 \\ 3a & 3 & -1 \end{bmatrix} = \begin{bmatrix} 0 & 3 & 3a \\ 2b & 1 & 3 \\ -2 & 3 & -1 \end{bmatrix}$$

Now, by equality of matrices, we get

$$2b = 3$$
$$\Rightarrow \quad b = \frac{3}{2}$$
and
$$3a = -2$$
$$\Rightarrow \quad a = \frac{-2}{3}$$

Therefore, $a = \frac{-2}{3}$ and $b = \frac{3}{2}$ **Ans.**

4. Find the position vector of a point which divides the join of points with position vectors $\vec{a} - 2\vec{b}$ and $2\vec{a} + \vec{b}$ externally in the ratio 2 : 1. [1]

Solution : Let A and B be the given points with position vectors $\vec{a} - 2\vec{b}$ and $2\vec{a} + \vec{b}$ respectively.

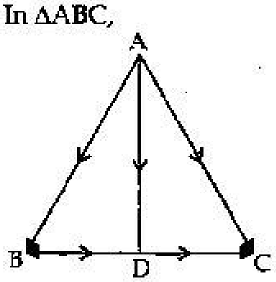
Let P be the point dividing AB in the ratio 2 : 1 externally.

$\therefore$ Position vector of $= \frac{2\times(2\vec{a}+\vec{b}) - 1\times(\vec{a}-2\vec{b})}{2-1}$

$= 3\vec{a} + 4\vec{b}$ **Ans.**

5. The two vectors $\hat{j} + \hat{k}$ and $3\hat{i} - \hat{j} + 4\hat{k}$ represent the two sides AB and AC, respectively of a ΔABC. Find the length of the median through A. [1]

Solution : In ΔABC,



Using the triangle law of vector addition, we have

$$\vec{BC} = \vec{AC} - \vec{AB}$$
$$= (3\hat{i} - \hat{j} + 4\hat{k}) - (\hat{j} + \hat{k})$$
$$= 3\hat{i} - 2\hat{j} + 3\hat{k}$$

$$\therefore \vec{BD} = \frac{1}{2}\vec{BC} = \frac{3}{2}\hat{i} - \hat{j} + \frac{3}{2}\hat{k}$$

(since AD is the median)

In ΔABD, using the triangle law of vector addition, we have

$$\vec{AD} = \vec{AB} + \vec{BD}$$
$$= (\hat{j} + \hat{k}) + \left(\frac{3}{2}\hat{i} - \hat{j} + \frac{3}{2}\hat{k}\right)$$
$$= \frac{3}{2}\hat{i} + 0\hat{j} + \frac{5}{2}\hat{k}$$

$$\therefore AD = \sqrt{\left(\frac{3}{2}\right)^2 + 0^2 + \left(\frac{5}{2}\right)^2} = \frac{1}{2}\sqrt{34}$$

Hence, the length of the median through A is $\frac{1}{2}\sqrt{34}$ units. **Ans.**

6. Find the vector equation of a plane which is at a distance of 5 units from the origin and its normal vector is $2\hat{i} - 3\hat{j} + 6\hat{k}$. [1]

Solution : Here, $d = 5$ units and $\vec{n} = 2\hat{i} - 3\hat{j} + 6\hat{k}$

$$\therefore \quad \hat{n} = \frac{\vec{n}}{|\vec{n}|} = \frac{2\hat{i} - 3\hat{j} + 6\hat{k}}{\sqrt{4+9+36}} = \frac{2\hat{i} - 3\hat{j} + 6\hat{k}}{\sqrt{49}}$$

$$\Rightarrow \quad \hat{n} = \frac{2}{7}\hat{i} - \frac{3}{7}\hat{j} + \frac{6}{7}\hat{k}$$

Hence, the required equation of the plane is

$$\vec{r}\cdot\left(\frac{2}{7}\hat{i} - \frac{3}{7}\hat{j} + \frac{6}{7}\hat{k}\right) = 5 \quad [\because \vec{r}\cdot\hat{n} = d]$$

or $\vec{r}\cdot(2\hat{i} - 3\hat{j} + 6\hat{k}) = 35$ **Ans.**

SECTION — B

7. Prove that :

$$\tan^{-1}\frac{1}{5} + \tan^{-1}\frac{1}{7} + \tan^{-1}\frac{1}{3} + \tan^{-1}\frac{1}{8} = \frac{\pi}{4}$$ **[4]**

Solution LHS =

$$\left(\tan^{-1}\frac{1}{5} + \tan^{-1}\frac{1}{7}\right) + \left(\tan^{-1}\frac{1}{3} + \tan^{-1}\frac{1}{8}\right)$$

$$= \tan^{-1}\left(\frac{\frac{1}{5}+\frac{1}{7}}{1-\frac{1}{5}\times\frac{1}{7}}\right) + \tan^{-1}\left(\frac{\frac{1}{3}+\frac{1}{8}}{1-\frac{1}{3}\times\frac{1}{8}}\right)$$

$$\left[\because \tan^{-1}A + \tan^{-1}B = \tan^{-1}\left(\frac{A+B}{1-AB}\right)\right]$$

$$= \tan^{-1}\frac{6}{17} + \tan^{-1}\frac{11}{23}$$

$$= \tan^{-1}\left(\frac{\frac{6}{17}+\frac{11}{23}}{1-\frac{6}{17}\times\frac{11}{23}}\right)$$

$$= \tan^{-1}\left(\frac{325}{325}\right)$$

$$= \tan^{-1}(1)$$

$$= \frac{\pi}{4}$$

Hence Proved.

OR

Solve for x :

$$\mathbf{2\tan^{-1}(\cos x) = \tan^{-1}(2\,\text{cosec}\,x)}$$

Solution : Given, $2\tan^{-1}(\cos x) = \tan^{-1}(2\text{cosec}\,x)$

$$\Rightarrow \quad \tan^{-1}\left(\frac{2\cos x}{1-\cos^2 x}\right) = \tan^{-1}(2\,\text{cosec}\,x)$$

$$\left[\because 2\tan^{-1}x = \tan^{-1}\left(\frac{2x}{1-x^2}\right)\right]$$

$\Rightarrow$ $\frac{2\cos x}{\sin^2 x} = 2 \operatorname{cosec} x = \frac{2}{\sin x}$

$\Rightarrow$ $\cos x = \sin x$

$\Rightarrow$ $\tan x = 1$

$\Rightarrow$ $x = \frac{\pi}{4}$ **Ans.**

8. The monthly incomes of Aryan and Babban are in the ratio 3 : 4 and their monthly expenditures are in the ratio 5 : 7. If each saves ₹ 15,000 per month, find their monthly incomes using matrix method. This problem reflects which value ? [4]

Solution : Let the monthly incomes of Aryan and Babban be $3x$ and $4x$ respectively.

Suppose their monthly expenditures are $5y$ and $7y$ respectively.

Since each saves ₹ 15,000 per month.

Monthly saving of Aryan : $3x - 5y = 15{,}000$

Monthly saving of Babban : $4x - 7y = 15{,}000$

The above system of equations can be written in the matrix form as follows :

$$\begin{bmatrix} 3 & -5 \\ 4 & -7 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 15000 \\ 15000 \end{bmatrix}$$

AX = B, where $A = \begin{bmatrix} 3 & -5 \\ 4 & -7 \end{bmatrix}, X = \begin{bmatrix} x \\ y \end{bmatrix}$ and $B = \begin{bmatrix} 15000 \\ 15000 \end{bmatrix}$

Now, $|A| = \begin{vmatrix} 3 & -5 \\ 4 & -7 \end{vmatrix} = -21-(-20) = -1 \neq 0$

$$\text{adj } A = \begin{bmatrix} -7 & -4 \\ 5 & 3 \end{bmatrix}^T = \begin{bmatrix} -7 & 5 \\ -4 & 3 \end{bmatrix}$$

So, $A^{-1} = \frac{1}{|A|}\text{adj } A = -1\begin{bmatrix} -7 & 5 \\ -4 & 3 \end{bmatrix} = \begin{bmatrix} 7 & -5 \\ 4 & -3 \end{bmatrix}$

$\therefore$ $x = A^{-1}B$

$\Rightarrow$ $\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 7 & -5 \\ 4 & -3 \end{bmatrix}\begin{bmatrix} 15000 \\ 15000 \end{bmatrix}$

$\Rightarrow$ $\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 105000-75000 \\ 60000-45000 \end{bmatrix}$

$\Rightarrow$ $\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 30000 \\ 15000 \end{bmatrix}$

$\Rightarrow x = 30{,}000$ and $y = 15{,}000$

Therefore,

Monthly income of Aryan = 3 × 30,000 = ₹ 90,000

Monthly income of Babban = 4 × 30,000 = ₹ 1,20,000

Value : Saving in good time helps us to survive in bad times. **Ans.**

9. If $x = a\sin 2t\,(1+\cos 2t)$ and $y = b\cos 2t\,(1-\cos 2t)$, find the values of $\frac{dy}{dx}$ at $t = \frac{\pi}{4}$ and $t = \frac{\pi}{3}$ [4]

Solution : We have, $x = a \sin 2t\,(1 + \cos 2t)$

$\Rightarrow$ $\frac{dx}{dt} = 2a\cos 2t\,(1+\cos 2t) - 2a\sin^2 2t$

and, $y = b\cos 2t\,(1-\cos 2t)$

$\Rightarrow$ $\frac{dy}{dt} = 2b\cos 2t \sin 2t - 2b\sin 2t\,(1-\cot 2t)$

$\therefore$ $\frac{dy}{dx} = \frac{dy}{dt}\times\frac{dt}{dx}$

$$= \frac{2b\cos 2t\sin 2t - 2b\sin 2t(1-\cos 2t)}{2a\cos 2t(1+\cos 2t) - 2a\sin^2 2t}$$

$$= \frac{b[\cos 2t\sin 2t - \sin 2t(1-\cos 2t)]}{a[\cos 2t(1+\cos 2t) - \sin^2 2t]}$$

Since we know that at $t = \frac{\pi}{4}$, $\sin 2t = 1$ and $\cos 2t = 0$,

So, $\left(\frac{dy}{dx}\right)_{t=\frac{\pi}{4}} = \frac{b[0.1-1(1-0)]}{a[0(1+0)-1^2]} = \frac{b}{a}$

Also, we know that at $t = \frac{\pi}{3}$, $\sin 2t = \frac{\sqrt{3}}{2}$ and $\cos 2t = -\frac{1}{2}$

So $\left(\frac{dy}{dx}\right)_{t=\frac{\pi}{3}} = \frac{b\left[\left(-\frac{1}{2}\right)\left(\frac{\sqrt{3}}{2}\right)-\left(\frac{\sqrt{3}}{2}\right)\left(1+\frac{1}{2}\right)\right]}{a\left[\left(-\frac{1}{2}\right)\left(1-\frac{1}{2}\right)-\left(\frac{\sqrt{3}}{2}\right)^2\right]}$

$$= \frac{b\left[-\frac{\sqrt{3}}{4}-\frac{3\sqrt{3}}{4}\right]}{a\left[-\frac{1}{4}-\frac{3}{4}\right]}$$

$$= \frac{b[-\sqrt{3}]}{a[-1]} = \sqrt{3}\left(\frac{b}{a}\right)$$ **Ans.**

OR

If $y = x^x$ prove that $\frac{d^2y}{dx^2} - \frac{1}{y}\left(\frac{dy}{dx}\right)^2 - \frac{y}{x} = 0.$

Solution : Given, $y = x^x$

Taking log on both sides, we get

$\log y = \log (x^x)$

$\Rightarrow \quad \log y = x \log x$

On differentiating w.r.t. x, we get

$$\frac{1}{y}\frac{dy}{dx} = \log x + x \times \frac{1}{x} = 1 + \log x$$

$$\Rightarrow \quad \frac{dy}{dx} = x^x(1 + \log x)$$

$$\Rightarrow \quad \frac{dy}{dx} = y(1 + \log x)$$

Again differentiating w.r.t. x, we get

$$\Rightarrow \quad \frac{d^2y}{dx^2} = (1 + \log x)\frac{dy}{dx} + y\frac{d}{dx}(1 + \log x)$$

$$\Rightarrow \quad \frac{d^2y}{dx^2} = (1 + \log x).x^x(1 + \log x) + x^x \times \frac{1}{x}$$

$$\Rightarrow \quad \frac{d^2y}{dx^2} = x^x(1 + \log x)^2 + x^{x-1}$$

$$\Rightarrow \quad \frac{d^2y}{dx^2} = \frac{1}{y}\left(\frac{dy}{dx}\right)^2 + \frac{y}{x}$$

$$\Rightarrow \quad \frac{d^2y}{dx^2} - \frac{1}{y}\left(\frac{dy}{dx}\right)^2 - \frac{y}{x} = 0$$

Hence Proved.

10. Find the values of p and q for which

$$f(x) = \begin{cases} \dfrac{1-\sin^3 x}{3\cos^2 x}, & \text{if } x < \dfrac{\pi}{2} \\ p, & \text{if } x = \dfrac{\pi}{2} \\ \dfrac{q(1-\sin x)}{(\pi-2x)^2}, & \text{if } x > \dfrac{\pi}{2} \end{cases}$$ **[4]**

is continuous at $x = \pi/2$.

Solution: Given, $f(x) = \begin{cases} \dfrac{1-\sin^3 x}{3\cos^2 x}, & \text{if } x < \dfrac{\pi}{2} \\ p, & \text{if } x = \dfrac{\pi}{2} \\ \dfrac{q(1-\sin x)}{(\pi-2x)^2}, & \text{if } x > \dfrac{\pi}{2} \end{cases}$

$f(x)$ is continuous at $x = \frac{\pi}{2}$, then LHL = RHL = $f(a)$

i.e. $\lim\limits_{x\to\frac{\pi}{2}^-} f(x) = \lim\limits_{x\to\frac{\pi}{2}^+} f(x) = f\left(\frac{\pi}{2}\right)$

$$\lim_{x\to\frac{\pi}{2}^-} f(x) = \lim_{x\to\frac{\pi}{2}} \left(\frac{1-\sin^3 x}{3\cos^2 x}\right)$$

$$= \lim_{x\to\frac{\pi}{2}} \frac{(1-\sin x)(1+\sin^2 x+\sin x)}{3[1-\sin^2 x]}$$

$$= \lim_{x\to\frac{\pi}{2}} \frac{(1-\sin x)(1+\sin^2 x+\sin x)}{3(1+\sin x)(1-\sin x)}$$

$$\lim_{x\to\frac{\pi}{2}^-} f(x) = \lim_{x\to\frac{\pi}{2}} \frac{1-\sin^2 x+\sin x}{3(1+\sin x)} = \frac{1+1+1}{3(2)} = \frac{1}{2}$$

Let $x = \frac{\pi}{2} + \theta$ as $x \to \frac{\pi}{2}$, $\theta \to 0$

$$\lim_{x\to\frac{\pi}{2}^+} f(x) = \lim_{\theta\to 0} q\frac{\left[1-\sin\left(\frac{\pi}{2}-\theta\right)\right]}{(2\theta)^2} = \frac{q}{4}\lim_{\theta\to 0}\frac{1-\cos\theta}{\theta^2}$$

$$= \frac{q}{4}\lim_{\theta\to 0}\frac{2\sin^2\frac{\theta}{2}}{\theta^2} = \frac{q}{2}\lim_{\theta\to 0}\frac{\sin^2\frac{\theta}{2}}{4\times\left(\frac{\theta}{2}\right)^2} = \frac{q}{8}$$

Now, $\lim\limits_{x\to\frac{\pi}{2}^-} f(x) = \lim\limits_{x\to\frac{\pi}{2}^+} f(x) = f\left(\frac{\pi}{2}\right)$

$$\Rightarrow \quad \frac{1}{2} = p = \frac{q}{8}$$

$$\Rightarrow \quad p = \frac{1}{2} \text{ and } q = 4$$ **Ans.**

11. Show that the equation of normal at any point on the curve $x = 3\cos t - \cos^3 t$ and $y = 3\sin t - \sin^3$ *is* $4(y\cos^3 t - x\sin^3 t) = 3\sin 4t. - \sin^3 t$ is **[4]**

Solution : Given,

$$x = 3\cos t - \cos^3 t$$

$$\Rightarrow \quad \frac{dy}{dt} = -3\sin t + 3\cos^2 t \sin t$$

and $\quad y = 3\sin t - \sin^3 t$

$$\Rightarrow \quad \frac{dy}{dt} = 3\cos t - 3\sin^2 t\cos t$$

Slope of the tangent,

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{3\cos t - 3\sin^2 t\cos t}{-3\sin t + 3\cos^2 t\sin t}$$

$$= \frac{3\cos t[\cos^2 t]}{-3\sin t[\sin^2 t]}$$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{-\cos^3 t}{\sin^3 t}$$

$\therefore$ Slope of the normal $= \frac{-dy}{dx} = \frac{\sin^3 t}{\cos^3 t}$

The equation of the normal is given by

$$\frac{y-(3\sin t - \sin^3 t)}{x-(3\cos t - \cos^3 t)} = \frac{\sin^3 t}{\cos^3 t}$$

$$\Rightarrow \quad y\cos^3 t - 3\sin t\cos^3 t + \sin^3 t\cos^3 t$$
$$= x\sin^3 t - 3\cos t\sin^3 t + \sin^3 t\cos^3 t$$

$\Rightarrow\ y\cos^3 t - x\sin^3 t = 3\ (\sin t\cos^3 t - \cos t\sin^3 t)$

$\Rightarrow\ y\cos^3 t - x\sin^3 t = 3\sin t\cos t\ (\cos^2 t - \sin^2 t)$

$\Rightarrow\ y\cos^3 t - x\sin^3 t = \dfrac{3\times 2}{2}\ \sin t\cos t\cos 2t$

$\Rightarrow\ y\cos^3 t - x\sin^3 t = \dfrac{3}{2}\sin 2t\cos 2t$

$\Rightarrow\ y\cos^3 t - x\sin^3 t = \dfrac{3\times 2}{2\times 2}\ \sin 2t\cos 2t$

$\Rightarrow\ y\cos^3 t - x\sin^3 t = \dfrac{3}{4}\sin 4t$

$\Rightarrow 4\,(y\cos^3 t - x\sin^3 t) = 3\sin 4t$ **Hence Proved.**

12. Find $\displaystyle\int \frac{(3\sin\theta - 2)\cos\theta}{5-\cos^2\theta - 4\sin\theta}d\theta$ **[4]**

Solution : Let $\displaystyle I = \int \frac{(3\sin\theta-2)\cos\theta}{5-\cos^2\theta-4\sin\theta}.d\theta$

$\Rightarrow\quad \displaystyle I = \int \frac{(3\sin\theta-2)\cos\theta}{4+1-\cos^2\theta-4\sin\theta}d\theta$

$\Rightarrow\quad \displaystyle I = \int \frac{(3\sin\theta-2)\cos\theta.d\theta}{4+\sin^2\theta-4\sin\theta}$

$[\because 1-\cos^2\theta = \sin^2\theta]$

$\Rightarrow\quad \displaystyle I = \int \frac{(3\sin\theta-2)\cos\theta.d\theta}{(\sin\theta-2)^2}$

Put $\sin\theta = t$

$\Rightarrow\quad \cos\theta.\,d\theta = dt$

$\therefore\quad \displaystyle I = \int \frac{(3t-2)}{(t-2)^2}dt$

Consider,

$$\frac{3t-2}{(t-2)^2} = \frac{A}{(t-2)} + \frac{B}{(t-2)^2}$$

$$3t-2 = A\,(t-2) + B$$

On comparing, we get

$$A = 3 \text{ and } B = 4$$

$$I = \int\left(\frac{3}{t-2} + \frac{4}{(t-2)^2}\right)dt$$

$\Rightarrow\quad \displaystyle I = 3\log|t-2| - \frac{4}{(t-2)} + c$

$\Rightarrow\quad \displaystyle I = 3\log|\sin\theta-2| - \frac{4}{(\sin\theta-2)} + c$

Ans.

OR

Evaluate $\displaystyle\int_0^{\pi} e^{2x}.\sin\left(\frac{\pi}{4}+x\right)dx$

Solution : Let $\displaystyle I = \int_0^{\pi} e^{2x}.\sin\left(\frac{\pi}{4}+x\right)dx$

$$I = \left[\sin\left(\frac{\pi}{4}+x\right).\frac{e^{2x}}{2}\right]_0^{\pi} - \int_0^{\pi}\cos\left(\frac{\pi}{4}+x\right).\frac{e^{2x}}{2}dx$$

$\Rightarrow\ \displaystyle I = \sin\frac{5\pi}{4}.\frac{e^{2\pi}}{2} - \sin\frac{\pi}{4}\times\frac{1}{2} - \frac{1}{2}$

$$\left[\cos\left(\frac{\pi}{4}+x\right).\frac{e^{2x}}{2}\right]_0^{\pi} + \frac{1}{2}\int_0^{\pi} -\sin\left(\frac{\pi}{4}+x\right).\frac{e^{2x}}{2}dx$$

$\Rightarrow\ \displaystyle I = \frac{1}{2}\sin\frac{5\pi}{4}.e^{2\pi} - \frac{1}{2}\sin\frac{\pi}{4} - \frac{1}{4}$

$$\left[\cos\frac{5\pi}{4}.e^{2\pi} - \cos\frac{\pi}{4}\right] - \frac{1}{2}\times\frac{1}{2}\int_0^{\pi}\sin\left(\frac{\pi}{4}+x\right)e^{2x}.dx$$

$\Rightarrow\ \displaystyle I = \frac{1}{2}\left(-\frac{1}{\sqrt{2}}\right)e^{2\pi} - \frac{1}{2}\times\frac{1}{\sqrt{2}} - \frac{1}{4}\left(\frac{-1}{\sqrt{2}}e^{2\pi} - \frac{1}{\sqrt{2}}\right) - \frac{1}{4}I$

$\Rightarrow\ \displaystyle I + \frac{I}{4} = -\frac{1}{2\sqrt{2}}e^{2\pi} - \frac{1}{2\sqrt{2}} + \frac{1}{4\sqrt{2}}e^{2\pi} + \frac{1}{4\sqrt{2}}$

$\Rightarrow\ \displaystyle \frac{5I}{4} = \frac{-2e^{2\pi}-2+e^{2\pi}+1}{4\sqrt{2}} = \frac{-e^{2\pi}-1}{4\sqrt{2}}$

$\Rightarrow\ \displaystyle 5I = -\left(\frac{e^{2\pi}+1}{\sqrt{2}}\right)$

$\Rightarrow\ \displaystyle I = -\frac{1}{5}\left(\frac{e^{2\pi}+1}{\sqrt{2}}\right)$ **Ans.**

13. Find $\displaystyle\int\frac{\sqrt{x}}{\sqrt{a^3-x^3}}dx.$ **[4]**

Solution : Let $\displaystyle I = \int\frac{\sqrt{x}}{\sqrt{a^3-x^3}}dx$

Put $x^{3/2} = t$

$\Rightarrow\quad \dfrac{3}{2}\sqrt{x}.dx = dt$

$\Rightarrow\quad \sqrt{x}.dx = \dfrac{2}{3}dt$

Putting the values in I, we get

$$I = \frac{2}{3}\int\frac{1}{\sqrt{(a^{3/2})^2 - t^2}}dt$$

$\Rightarrow\quad \displaystyle I = \frac{2}{3}\sin^{-1}\left(\frac{t}{a^{3/2}}\right) + c$

or $\quad \displaystyle I = \frac{2}{3}\sin^{-1}\left(\frac{x}{a}\right)^{3/2} + c$ **Ans.**

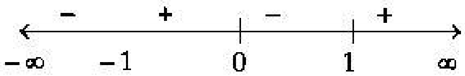
14. Evaluate : $\displaystyle\int_{-1}^{2}|x^3-x|\,dx.$ **[4]**

Solution : Let $\displaystyle I = \int_{-1}^{2}|x^3-x|\,dx.$

$$f(x) = x^3 - x$$

$$f(x) = x^3 - x = x(x-1)(x+1)$$

The signs of $f(x)$ for the different values are shown in the figure given below :



$$f(x) > 0 \text{ for all } x \in (-1, 0) \cup (1, 2)$$

$$f(x) < 0 \text{ for all } x \in (0, 1)$$

Therefore,

$$|x^3 - x| = \begin{cases} x^3 - x, & x \in (-1, 0) \cup (1, 2) \\ -(x^3 - x), & x \in (0, 1) \end{cases}$$

$$\therefore I = \int_{-1}^{2} |x^3 - x|\, dx$$

$$= \int_{-1}^{0} |x^3 - x|\, dx + \int_{0}^{1} |x^3 - x|\, dx + \int_{1}^{2} |x^3 - x|\, dx$$

$$= \int_{-1}^{0} (x^3 - x) dx - \int_{0}^{1} (x^3 - x) dx + \int_{1}^{2} (x^3 - x) dx$$

$$= \left[\frac{x^4}{4} - \frac{x^2}{2}\right]_{-1}^{0} - \left[\frac{x^4}{4} - \frac{x^2}{2}\right]_{0}^{1} + \left[\frac{x^4}{4} - \frac{x^2}{2}\right]_{1}^{2}$$

$$= -\left(\frac{1}{4} - \frac{1}{2}\right) - \left(\frac{1}{4} - \frac{1}{2}\right) + \left(\frac{16}{4} - \frac{4}{2}\right) - \left(\frac{1}{4} - \frac{1}{2}\right)$$

$$= \frac{3}{4} + (4 - 2)$$

$$= \frac{11}{4}$$

Ans.

15. Find the particular solution of the differential equation $(1 - y^2)(1 + \log x) dx + 2xy\, dy = 0$, given that $y = 0$ when $x = 1$. **[4]**

Solution : The given differential equation is,

$$(1 - y^2)(1 + \log x).dx + 2xy\, dy = 0$$

$$\Rightarrow \frac{(1 + \log x)}{x} dx = \frac{-2y}{(1 - y^2)} dy$$

On integrating both side, we have

$$\Rightarrow \int \frac{1 + \log x}{x} dx = \int \frac{-2y}{(1 - y^2)} dy$$

In first integral,

put $\quad 1 + \log x = t$

$$\Rightarrow \frac{1}{x} dx = dt$$

Also, in second integral,

put $\quad 1 - y^2 = u$

$$\Rightarrow -2y.dy = du$$

$$\therefore \int t.dt = \int \frac{1}{u}.du$$

$$\Rightarrow \frac{t^2}{2} - \log|u| = c$$

or $\frac{1}{2}(1 + \log x)^2 - \log|1 - y^2| = c$

It is given that $y = 0$ when $x = 1$

So, $\frac{1}{2}(1 + \log 1)^2 - \log|1 - 0^2| = c$

$$\Rightarrow c = \frac{1}{2}$$

$$\therefore \frac{(1 + \log x)^2}{2} - \log|1 - y^2| = \frac{1}{2}$$

or $(1 + \log x)^2 - 2\log|1 - y^2| = 1$

It is the required particular solution. **Ans.**

16. Find the general solution of the following differential equation :

$$(1 + y^2) + (x - e^{\tan^{-1} y}) \frac{dy}{dx} = 0$$ **[4]**

Solution : The given differential equation is,

$$(1 + y^2) + (x - e^{\tan^{-1} y}) \frac{dy}{dx} = 0$$

$$\Rightarrow \frac{dx}{dy} + \left(\frac{1}{1 + y^2}\right) x = \frac{e^{\tan^{-1} y}}{1 + y^2}$$

This is a linear differential equation with

$$P = \frac{1}{1 + y^2}$$

and

$$Q = \frac{e^{\tan^{-1} y}}{1 + y^2}$$

$$\therefore \text{I.F.} = e^{\int \frac{1}{1 + y^2} dx} = e^{\tan^{-1} y}$$

So, the required solution is :

$$x e^{\tan^{-1} y} = \int e^{\tan^{-1} y}. \frac{e^{\tan^{-1} y}}{1 + y^2} dy$$

Put $\quad \tan^{-1} y = t$

$$\Rightarrow \frac{1}{1 + y^2} dy = dt$$

$$\therefore x e^{\tan^{-1} y} = \int e^{2t}.dt$$

$$\Rightarrow x e^{\tan^{-1} y} = \frac{1}{2} e^{2t} + C$$

$$\Rightarrow x e^{\tan^{-1} y} = \frac{1}{2} e^{2\tan^{-1} y} + C$$

$$\Rightarrow x = \frac{1}{2} e^{\tan^{-1} y} - C e^{-\tan^{-1} y}$$ **Ans.**

17. Show that the vectors $\vec{a}, \vec{b}$ and $\vec{c}$ are coplanar if $\vec{a} + \vec{b}, \vec{b} + \vec{c}$ and $\vec{c} + \vec{a}$ are coplanar. **[4]**

Solution : As $\vec{a}+\vec{b}, \vec{b}+\vec{c}, \vec{c}+\vec{a}$ are coplanar.

So, $[\vec{a}+\vec{b} \quad \vec{b}+\vec{c} \quad \vec{c}+\vec{a}]=0$ ...(i)

Now, if $\vec{a}, \vec{b}$ and $\vec{c}$ are coplanar vectors

then, $[\vec{a} \quad \vec{b} \quad \vec{c}]=0$

Consider $[\vec{a}+\vec{b} \quad \vec{b}+\vec{c} \quad \vec{c}+\vec{a}]$

$=[(\vec{a}+\vec{b})\times(\vec{b}+\vec{c})].(\vec{c}+\vec{a})$

$=[\vec{a}\times\vec{b}+\vec{a}\times\vec{c}+\vec{b}\times\vec{b}+\vec{b}\times\vec{c}].(\vec{c}+\vec{a})$

$=(\vec{a}\times\vec{b}).\vec{c}+(\vec{a}\times\vec{b}).\vec{a}+(\vec{a}\times\vec{c}).\vec{c}+(\vec{a}\times\vec{c}).\vec{a}$
$+(\vec{b}\times\vec{c}).\vec{c}+(\vec{b}\times\vec{c}).\vec{a}$

$=(\vec{a}\times\vec{b}).\vec{c}+0+0+0+0+(\vec{b}\times\vec{c}).\vec{a}$

$=[\vec{a} \quad \vec{b} \quad \vec{c}]+[\vec{b} \quad \vec{c} \quad \vec{a}]$

$=[\vec{a} \quad \vec{b} \quad \vec{c}]+[\vec{a} \quad \vec{b} \quad \vec{c}]$

$=2[\vec{a} \quad \vec{b} \quad \vec{c}]$

From equation (i)

$=2[\vec{a} \quad \vec{b} \quad \vec{c}] = [\vec{a}+\vec{b} \quad \vec{b}+\vec{c} \quad \vec{c}+\vec{a}]=0$

$\Rightarrow \quad [\vec{a} \quad \vec{b} \quad \vec{c}] = 0$

$\therefore$ $\vec{a}, \vec{b}$ and $\vec{c}$ are coplanar if $\vec{a}+\vec{b}, \vec{b}+\vec{c}$ and $\vec{c}+\vec{a}$ are coplanar. **Hence Proved**

18. Find the vector and cartesian equations of the line through the point (1, 2, – 4) and perpendicular to the two lines.

$$\vec{r}=(8\hat{i}-19\hat{j}+10\hat{k})+\lambda(3\hat{i}-16\hat{j}+7\hat{k})$$

and $\vec{r}=(15\hat{i}+29\hat{j}-5\hat{k})+\mu(3\hat{i}+8\hat{j}-5\hat{k})$

Solution : The equations of the given lines are

$\vec{r}=(8\hat{i}-19\hat{j}+10\hat{k})+\lambda(3\hat{i}-16\hat{j}+7\hat{k})$ ......(i)

$\vec{r}=(15\hat{i}+29\hat{j}+5\hat{k})+\mu(3\hat{i}+8\hat{j}-5\hat{k})$ .....(ii)

Normal parallel to (i) is

$\vec{n_1}=3\hat{i}-16\hat{j}+7\hat{k}$

Normal parallel to (ii) is

$\vec{n_2}=3\hat{i}+8\hat{j}-5\hat{k}$

The required line is perpendicular to the given lines. So the normal $\vec{n}$ parallel to the required line perpendicular to $\vec{n_1}$ and $\vec{n_2}$.

$$\therefore \vec{n}=\vec{n_1}\times\vec{n_2}=\begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ 3 & -16 & 7\\ 3 & 8 & -5\end{vmatrix}=24\hat{i}+36\hat{j}+72\hat{k}$$

Thus, the vector equation of the required line is

$\vec{r}=(\hat{i}+2\hat{j}-4\hat{k})+\gamma(24\hat{i}+36\hat{j}+72\hat{k})$

$\Rightarrow \quad \vec{r}=(\hat{i}+2\hat{j}-4\hat{k})+k(2\hat{i}+3\hat{j}+6\hat{k})$

(where $k=12\gamma$)

Also, the cartesian equation of the required line is

$$\frac{x-1}{2}=\frac{y-2}{3}=\frac{z+4}{6}$$ **Ans.**

19. Three persons A, B and C apply for a job of Manager in a Private Company. Chances of their selection (A, B and C) are in the ratio 1 : 2 : 4. The probabilities that A, B and C can introduce changes to improve profits of the company are 0.8, 0.5 and 0.3 respectively. If the change does not take place, find the probability that it is due to the appointment of C. **[4]**

Solution : Let E_1, E_2 and E_3 be the events denoting the selection of A, B and C as managers respectively.

$P(E_1)$ = Probability of selection of A = $\frac{1}{7}$

$P(E_2)$ = Probability of selection of B = $\frac{2}{7}$

$P(E_3)$ = Probability of selection of C = $\frac{4}{7}$

Let A be the event denoting the change not taking place.

$P(A/E_1)$ = Probability that A does not introduce change = 0.2

$P(A/E_2)$ = Probability that B does not introduce change = 0.5

$P(A/E_3)$ = Probability that C does not introduce change = 0.7

$\therefore$ Required probability = $P(E_3/A)$

By Bayes' theorem, we have

$P(E_3/A)$ =

$$\frac{P(E_3)P(A/E_3)}{P(E_1)P(A/E_1)+P(E_2)P(A/E_2)+P(E_3)P(A/E_3)}$$

$$=\frac{\frac{4}{7}\times 0.7}{\frac{1}{7}\times 0.2+\frac{2}{7}\times 0.5+\frac{4}{7}\times 0.7}$$

$$= \frac{2.8}{0.2+1+2.8}$$

$$= \frac{2.8}{4} = 0.7$$ **Ans.**

OR

A and B throw a pair of dice alternately. A wins the game if he gets a total of 7 and B wins the game if he gets a total of 10. If A starts the game, then find the probability that B wins.

Solution : Total of 7 on the dice can be obtained in the following ways :

(1, 6), (6, 1), (2, 5), (5, 2), (3, 4), (4, 3)

Probability of getting a total of 7 = $\frac{6}{36} = \frac{1}{6}$

Probability of not getting a total of 7 = $1 - \frac{1}{6} = \frac{5}{6}$

Total of 10 on the dice can be obtained in the following ways :

(4, 6), (6, 4), (5, 5)

Probability of getting a total of 10

$$= \frac{3}{36} = \frac{1}{12}$$

Probability of not getting a total of 10

$$= 1 - \frac{1}{12} = \frac{11}{12}$$

Let E and F be the two events, defined as follows :

E = Getting a total of 7 in a single throw of a dice

F = Getting a total of 10 in a single throw of a dice

$$P(E) = \frac{1}{6}, P(\bar{E}) = \frac{5}{6}$$

$$P(F) = \frac{1}{12}, P(\bar{F}) = \frac{11}{12}$$

A wins if he gets a total of 7 in 1st, 3rd or 5th.... throws.

Probability of A getting a total of 7 in the1st throw $= \frac{1}{6}$

A will get the 3rd throw if he fails in the 1st throw and B fails in the 2nd throw.

Probability of A getting a total of 7 in the 3rd throw

$$= P(\bar{E})\, P(\bar{F})\, P(E) = \frac{5}{6} \times \frac{11}{12} \times \frac{1}{6}$$

Similarly, probability of getting a total of 7 in the 5th throw = $P(\bar{E})\, P(\bar{F})\, P(\bar{E})\, P(\bar{E})\, P(E)$

$$= \frac{5}{6} \times \frac{11}{12} \times \frac{5}{6} \times \frac{11}{12} \times \frac{1}{6} \text{ and so on}$$

Probability of winning of A

$$= \frac{1}{6} + \left(\frac{5}{6} \times \frac{11}{12} \times \frac{1}{6}\right) + \left(\frac{5}{6} \times \frac{11}{12} \times \frac{5}{6} \times \frac{11}{12} \times \frac{1}{6}\right) + \ldots$$

$$= \frac{\frac{1}{6}}{1 - \frac{5}{6} \times \frac{11}{12}} = \frac{12}{17}$$

$\therefore$ Probability of winning of B = 1 – Probability of winning of A

$$= 1 - \frac{12}{17} = \frac{5}{17}$$ **Ans.**

SECTION — C

20. Let $f : N \to N$ be a function defined as $f(x) = 9x^2 + 6x - 5$. Show that $f : N \to S$, where S is the range of f, is invertible. Find the inverse of f and hence find $f^{-1}(43)$ and $f^{-1}(163)$. [6]

Solution : Given,

$$f(x) = 9x^2 + 6x - 5$$

Let $x_1, x_2 \in N$ and $f(x_1) = f(x_2)$

$\Rightarrow \quad 9x_1^2 + 6x_1 - 5 = 9x_2^2 + 6x_2 - 5$

$\Rightarrow \quad 9(x_1^2 - x_2^2) + 6(x_1 - x_2) = 0$

$\Rightarrow \quad (x_1 - x_2)[9(x_1 + x_2) + 6] = 0$

$\Rightarrow \quad x_1 - x_2 = 0$

as $\quad 9x_1 + 9x_2 + 6 \neq 0 \quad [x_1, x_2 \in N]$

$\Rightarrow \quad x_1 = x_2$

f is one-one function.

Let $\quad y = 9x^2 + 6x - 5$

$\Rightarrow \quad y = (3x+1)^2 - 1 - 5 = (3x+1)^2 - 6$

$\Rightarrow \quad (3x+1)^2 = y + 6$

$\Rightarrow \quad 3x + 1 = \sqrt{y+6}$

$\Rightarrow \quad x = \frac{\sqrt{y+6} - 1}{3}$ as $x \in N$

$\Rightarrow \quad \sqrt{y+6} - 1 > 0$

$\Rightarrow \quad y + 6 > 1$

$\Rightarrow \quad y > -5$ and $y \in N$

So, the function is invertible if the range of the function $f(x)$ is {1, 2, 3, ...}.

$f : N \to S$ is onto as codomain = Range

Hence f is invertible.

Therefore, the inverse of the function $f(x)$ is $f^{-1}(y)$, *i.e.*, x.

Now, $\quad f^{-1}(y) = \frac{\sqrt{y+6} - 1}{3}$

$$\Rightarrow \qquad f^{-1}(43) = \frac{\sqrt{43+6}-1}{3} = 2$$

$$\text{and} \qquad f^{-1}(163) = \frac{\sqrt{163+6}-1}{3} = 4 \qquad \textbf{Ans.}$$

21. Prove that $\begin{vmatrix} yz-x^2 & zx-y^2 & xy-z^2 \\ zx-y^2 & xy-z^2 & yz-x^2 \\ xy-z^2 & yz-x^2 & zx-y^2 \end{vmatrix}$ is visible by $(x+y+z)$ and hence find the quotient. **[6]**

Solution : Let $\Delta = \begin{vmatrix} yz\ \ x^2 & zx-y^2 & xy-z^2 \\ zx-y^2 & xy-z^2 & yz-x^2 \\ xy-z^2 & yz-x^2 & zx-y^2 \end{vmatrix}$

Applying $C_1 \to C_1 - C_2$, $C_2 \to C_2 - C_3$

$$\Delta = \begin{vmatrix} yz-x^2-zx+y^2 & zx-xy-y^2+z^2 & xy-z^2 \\ zx-xy-y^2+z^2 & xy-yz-z^2+x^2 & yz-x^2 \\ xy-yz-z^2+x^2 & yz-zx-x^2+y^2 & zx-y^2 \end{vmatrix}$$

$$\Delta = \begin{vmatrix} (y-x)(x+y+z) & (z-y)(x+y+z) & xy-z^2 \\ (z-y)(x+y+z) & (x-z)(x+y+z) & yz-x^2 \\ (x-z)(x+y+z) & (y-x)(x+y+z) & zx-y^2 \end{vmatrix}$$

Taking $(x+y+z)$ common from C_1 and C_2 both

$$\Delta = (x+y+z)^2 \begin{vmatrix} (y-x) & (z-y) & xy-z^2 \\ (z-y) & (x-z) & yz-x^2 \\ (x-z) & (y-x) & zx-y^2 \end{vmatrix}$$

Applying $R_1 \to R_1 + R_2 + R_3$

$$\Delta = (x+y+z)^2 \begin{vmatrix} 0 & 0 & xy+yz+zx-x^2-y^2-z^2 \\ (z-y) & (x-y) & yz-x^2 \\ (x-z) & (y-x) & zx-y^2 \end{vmatrix}$$

Expanding along R_1, we get

$$\Delta = (x+y+z)^2 \{(xy+yz+zx-x^2-y^2-z^2)\ [(z-y)(y-x)-(x-z)^2]\}$$

$$\Rightarrow \Delta = (x+y+z)^2 \{(xy+yz+zx-x^2-y^2-z^2)\ (xy+yz+zx-x^2-y^2-z^2)\}$$

$$\Delta = (x+y+z)^2 (xy+zy+zx-x^2-y^2-z^2)^2$$

Hence Δ is divisible by $(x+y+z)$ and the quotient is $(x+y+z)(xy+yz+zx-x^2-y^2-z^2)^2$

Hence Proved.

OR

Using elementary transformations, find the inverse of the matrix $A = \begin{bmatrix} 8 & 4 & 3 \\ 2 & 1 & 1 \\ 1 & 2 & 2 \end{bmatrix}$ and use it to solve the following system of linear equations :

$$8x + 4y + 3z = 19$$
$$2x + y + z = 5$$
$$x + 2y + 2z = 7$$

Solution: Here, $A = \begin{bmatrix} 8 & 4 & 3 \\ 2 & 1 & 1 \\ 1 & 2 & 2 \end{bmatrix}$

Using $A = IA$, we have

$$\begin{bmatrix} 8 & 4 & 3 \\ 2 & 1 & 1 \\ 1 & 2 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} A$$

Applying $R_1 \leftrightarrow R_3$, We get

$$\begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & 1 \\ 8 & 4 & 3 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} A$$

Applying $R_2 \to R_2 - 2R_1$, $R_3 \to R_3 - 8R_1$, We get

$$\begin{bmatrix} 1 & 2 & 2 \\ 0 & -3 & -3 \\ 0 & -12 & -13 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & -2 \\ 1 & 0 & -8 \end{bmatrix} A$$

Applying $R_2 \to \frac{R_2}{-3}$, we get

$$\begin{bmatrix} 1 & 2 & 2 \\ 0 & 1 & 1 \\ 0 & -12 & -13 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & -\frac{1}{3} & \frac{2}{3} \\ 1 & 0 & -8 \end{bmatrix} A$$

Applying $R_1 \to R_1 - 2R_2$, we get

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & -12 & -13 \end{bmatrix} = \begin{bmatrix} 0 & \frac{2}{3} & -\frac{1}{3} \\ 0 & -\frac{1}{3} & \frac{2}{3} \\ 1 & 0 & -8 \end{bmatrix} A$$

Applying $R_3 \to R_3 + 12R_2$, we get

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & -1 \end{bmatrix} = \begin{bmatrix} 0 & \frac{2}{3} & -\frac{1}{3} \\ 0 & -\frac{1}{3} & \frac{2}{3} \\ 1 & -4 & 0 \end{bmatrix} A$$

Applying $R_3 \to -R_3$, we get

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & \frac{2}{3} & -\frac{1}{3} \\ 0 & -\frac{1}{3} & \frac{2}{3} \\ -1 & 4 & 0 \end{bmatrix} A$$

Applying $R_2 \to R_2 - R_3$, we get

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & \frac{2}{3} & -\frac{1}{3} \\ 1 & -\frac{13}{3} & \frac{2}{3} \\ -1 & 4 & 0 \end{bmatrix} A$$

Thus, we have

$$A^{-1} = \begin{bmatrix} 0 & \frac{2}{3} & -\frac{1}{3} \\ 1 & -\frac{13}{3} & \frac{2}{3} \\ -1 & 4 & 0 \end{bmatrix}$$

The given system of equations is

$$8x + 4y + 3z = 19$$
$$2x + y + z = 5$$
$$x + 2y + 2z = 7$$

This system of equations can be written as $AX = B$,

where $A = \begin{bmatrix} 8 & 4 & 3 \\ 2 & 1 & 1 \\ 1 & 2 & 2 \end{bmatrix}, X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ and $B = \begin{bmatrix} 19 \\ 5 \\ 7 \end{bmatrix}$

$\therefore \quad X = A^{-1}B$

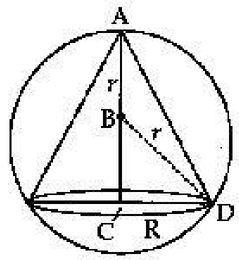
$$\Rightarrow \quad \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 & \frac{2}{3} & -\frac{1}{3} \\ 1 & -\frac{13}{3} & \frac{2}{3} \\ -1 & 4 & 0 \end{bmatrix} \begin{bmatrix} 19 \\ 5 \\ 7 \end{bmatrix}$$

$$\Rightarrow \quad \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 + \frac{10}{3} - \frac{7}{3} \\ 19 - \frac{65}{3} + \frac{14}{3} \\ -19 + 20 + 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$$

$\therefore x = 1, y = 2$ and $z = 1$. **Ans.**

22. Show that the altitude of the right circular cone of maximum volume that can be inscribed in a sphere of radius r is $\frac{4r}{3}$. Also find maximum volume in terms of volume of the sphere. [6]

Solution : A sphere of fixed radius (r) is given. Let R and h be the radius and the height of the cone respectively.



The volume (V) of the cone is given by,

$$V = \frac{1}{3}\pi R^2 h$$

Now, from the right triangle BCD, we have

$$BC = \sqrt{r^2 - R^2}$$

$$\therefore \quad h = r + \sqrt{r^2 - R^2}$$

$$V = \frac{1}{3}\pi R^2 r + \frac{1}{3}\pi R^2 \sqrt{r^2 - R^2}$$

$$\frac{dV}{dR} = \frac{2}{3}\pi R r + \frac{2}{3}\pi R \sqrt{r^2 - R^2} + \frac{\pi R^2}{3} \cdot \frac{(-2R)}{2\sqrt{r^2 - R^2}}$$

$$= \frac{2}{3}\pi R r + \frac{2}{3}\pi R \sqrt{r^2 - R^2} - \frac{\pi R^3}{3\sqrt{r^2 - R^2}}$$

$$= \frac{2}{3}\pi R r + \frac{2\pi R(r^2 - R^2) - \pi R^3}{3\sqrt{r^2 - R^2}}$$

$$= \frac{2}{3}\pi R r + \frac{2\pi R r^2 - 3\pi R^3}{3\sqrt{r^2 - R^2}}$$

Now, $\frac{dV}{dR} = 0$

$$\Rightarrow \quad \frac{2\pi r R}{3} = \frac{3\pi R^3 - 2\pi R r^2}{3\sqrt{r^2 - R^2}}$$

$$\Rightarrow \quad 2r\sqrt{r^2 - R^2} = 3R^2 - 2r^2$$

$$\Rightarrow \quad 4r^2(r^2 - R^2) = (3R^2 - 2r^2)^2$$

$$\Rightarrow \quad 4r^4 - 4r^2R^2 = 9R^4 + 4r^4 - 12R^2r^2$$

$$\Rightarrow \quad 9R^4 - 8r^2R^2 = 0$$

$$\Rightarrow \quad 9R^2 = 8r^2$$

$$\Rightarrow \quad R^2 = \frac{8r^2}{9}$$

Now, $\frac{d^2V}{dR^2}$

$$= \frac{2\pi r}{3} + \frac{3\sqrt{r^2 - R^2}(2\pi r^2 - 9\pi R^2) - + (2\pi R r^2 - 3\pi R^3)(-6R)\frac{1}{2\sqrt{r^2 - R^2}}}{9(r^2 - R^2)}$$

$$-\frac{2\pi r}{3}+\frac{3\sqrt{r^2-R^2}(2\pi r^2-9\pi R^2)}{\dfrac{(2\pi Rr^2-3\pi R^3)(6R)-\frac{1}{2\sqrt{r^2-R^2}}}{9(r^2-R^2)}}$$

No, when $R^2=\frac{8r^2}{9}$, Clearly $\frac{d^2V}{dR^2}<0.$

$\therefore$ The volume is maximum when

$$R^2=\frac{8r^2}{9}.$$

Height of the cone

$$=r+\sqrt{r^2-\frac{8r^2}{9}}=r+\sqrt{\frac{r^2}{9}}=r+\frac{r}{3}=\frac{4r}{3}.$$

Hence, it can be seen that the altitude of a right circular cone of maximum volume that can be inscribed in a sphere of radius r is $\frac{4r}{3}$.

Let volume of the sphere be $V_s=\frac{4}{3}\pi r^3$.

$$r=\sqrt[3]{\frac{3V_s}{4\pi}}$$

$\therefore$ Volume of cone, $V=\frac{1}{3}\pi R^2 h$

$$\Rightarrow \quad V=\frac{1}{3}\pi\frac{8r^2}{9}\times\frac{4r}{3} \qquad \left(\because R^2=\frac{8r^2}{9}\right)$$

$$\Rightarrow \quad V=\frac{32\pi r^3}{81}=\frac{8}{27}\left(\frac{4}{3}\pi r^3\right)$$

$\therefore$ Maximum volume of cone in terms of sphere = $\frac{8 \text{ Volume of sphere}}{27}$ **Hence Proved.**

OR

Find the intervals in which $f(x)=\sin 3x-\cos 3x$, $0<x<\pi$, is strictly increasing or strictly decreasing.

Solution : Consider the function

$$f(x)=\sin 3x-\cos 3x$$

$$\Rightarrow f'(x)=3\cos 3x+3\sin 3x$$

$$=3(\sin 3x+\cos 3x)$$

$$=3\sqrt{2}\left\{\sin 3x\cos\left(\frac{\pi}{4}\right)+\cos 3x\sin\left(\frac{\pi}{4}\right)\right\}$$

$$=3\sqrt{2}\left\{\sin\left(3x+\frac{\pi}{4}\right)\right\}$$

For the increasing interval $f'(x)>0$.

$$3\sqrt{2}\left\{\sin\left(3x+\frac{\pi}{4}\right)\right\}>0$$

$$\sin\left(3x+\frac{\pi}{4}\right)>0$$

$$\Rightarrow \quad 0<3x+\frac{\pi}{4}<\pi$$

$$\Rightarrow \quad \frac{-\pi}{4}<3x<\frac{3\pi}{4}$$

$$\Rightarrow \quad \frac{-\pi}{12}<x<\frac{\pi}{4}$$

as $0<x<\pi$ is given

$\Rightarrow 0<x<\pi/4$

Also, $\sin\left(3x+\frac{\pi}{4}\right)>0$

when, $2\pi<3x+\frac{\pi}{4}<3\pi$

Therefore, intervals in which function is strictly increasing is $0<x<\frac{\pi}{4}$ and $\frac{7\pi}{12}<x<\frac{11\pi}{12}$.

Similarly, for the decreasing interval $f'(x)<0$

$$3\sqrt{2}\left\{\sin\left(3x+\frac{\pi}{4}\right)\right\}<0$$

$$\sin\left(3x+\frac{\pi}{4}\right)<0$$

$$\Rightarrow \quad \pi<3x+\frac{\pi}{4}<2\pi$$

$$\Rightarrow \quad \frac{3\pi}{4}<3x<\frac{7\pi}{4}$$

$$\Rightarrow \quad \frac{\pi}{4}<x<\frac{7\pi}{12}$$

Also, $\sin\left(3x+\frac{\pi}{4}\right)<0$

When $3\pi<3x+\frac{\pi}{4}<4\pi$,

$$\Rightarrow \quad \frac{11\pi}{4}<3x<\frac{15\pi}{4}$$

$$\Rightarrow \quad \frac{11\pi}{12}<x<\frac{15\pi}{12}$$

But $0<x<\pi$ so $\frac{11\pi}{12}<x<\pi$

The function is strictly decreasing in $\frac{\pi}{4}<x<\frac{7\pi}{12}$ and $\frac{11\pi}{12}<x<\pi$. **Ans.**

23. Using integration find the area of the region $\{(x, y): x^2+y^2\le 2ax, y^2\ge ax, x, y\ge 0\}$ **[6]**

Solution : We have, $\{(x, y): x^2+y^2\le 2ax, y^2\ge ax, x, y\ge 0\}$

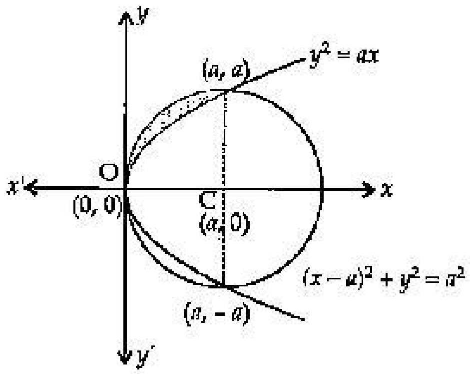
Consider $x^2+y^2 = 2ax$ ...(i)

$$y^2 = ax \quad ...(ii)$$
$$x = 0, y = 0$$

Solving equation (i) and (ii), we get

$$x^2 + ax = 2ax$$
$$\Rightarrow \quad x^2 - ax = 0$$
$$\Rightarrow \quad x(x-a) = 0$$
$$\therefore \quad x = 0, a$$

So, points of intersections of (i) and (ii) are (0, 0) and $(a, \pm a)$. Also, equation (i) can be written as, $(x-a)^2 + (y-0)^2 = a^2$ whose centre is at $(a, 0)$ and radius is of 'a' units.



$\therefore$ Require area

$$= \int_0^a y_1 dx - \int_0^a y_2 dx$$
$$= \int_0^a \sqrt{a^2-(x-a)^2}\,dx - \sqrt{a}\int_0^a \sqrt{x}\,dx$$
$$= \left[\frac{x-a}{2}\sqrt{a^2-(x-a)^2} + \frac{a^2}{2}\sin^{-1}\left(\frac{x-a}{a}\right)\right]_0^a - \frac{2}{3}\sqrt{a}[x^{3/2}]_0^a$$
$$= [0+0] - \left[0 + \frac{a^2}{2}\left(-\frac{\pi}{2}\right)\right] - \frac{2}{3}\sqrt{a}[a^{3/2} - 0]$$
$$= \left(\frac{\pi}{4} - \frac{2}{3}\right)a^2 \text{ sq. units}$$

Ans.

24. Find the coordinate of the point P where the line through A(3, – 4, – 5) and B (2, – 3, 1) crosses the plane passing through three points L (2, 2, 1), M (3, 0, 1) and N (4, – 1, 0). Also, find the ratio in which P divides the line segment AB. [6]

Solution : The equation of the plane passing through three given points can be given by

$$\begin{vmatrix} x-2 & y-2 & z-1 \\ x-3 & y-0 & z-1 \\ x-4 & y+1 & z-0 \end{vmatrix} = 0$$

Performing elementary row operations $R_2 \to R_1 - R_2$ and $R_3 \to R_1 - R_3$, we get

$$\begin{vmatrix} x-2 & y-2 & z-1 \\ 3-2 & 0-2 & 0 \\ 4-2 & -1-2 & -1 \end{vmatrix} = 0$$

$$\Rightarrow \begin{vmatrix} x-2 & y-2 & z-1 \\ 1 & -2 & 0 \\ 2 & -3 & -1 \end{vmatrix} = 0$$

Solving the above determinant, we get

$$(x-2)(2-0) - (y-2)(-1-0) + (z-1)(-3+4) = 0$$
$$\Rightarrow \quad (2x-4) + (y-2) + (z-1) = 0$$
$$\Rightarrow \quad 2x + y + z - 7 = 0$$

Therefore, the equation of the plane is

$$2x + y + z - 7 = 0$$

Now, the equation of the line passing through two given points is

$$\frac{x-3}{2-3} = \frac{y+4}{-3+4} = \frac{z+5}{1+5} = \lambda$$
$$\Rightarrow \quad \frac{x-3}{-1} = \frac{y+4}{1} = \frac{z+5}{6} = \lambda$$
$$\Rightarrow \quad x = (-\lambda+3), y = (\lambda-4), z = (6\lambda-5)$$

At the point of intersection, these points satisfy the equation of the plane $2x + y + z - 7 = 0$

Putting the values of x, y and z in the equation of the plane, we get the value of λ

$$2(-\lambda+3) + (\lambda-4) + (6\lambda-5) - 7 = 0$$
$$\Rightarrow \quad -2\lambda + 6 + \lambda - 4 + 6\lambda - 5 - 7 = 0$$
$$\Rightarrow \quad 5\lambda = 10$$
$$\Rightarrow \quad \lambda = 2$$

Thus, the point of intersection is P (1, – 2, 7).

Now, let P divide the line AB in the ratio $m : n$.

By the section formula, we have

$$1 = \frac{2m+3n}{m+n}$$
$$\Rightarrow \quad m + 2n = 0$$
$$\Rightarrow \quad m = -2n$$
$$\Rightarrow \quad \frac{m}{n} = \frac{-2}{1}$$

Hence, P divides externally the line segment AB in the ratio 2 : 1. **Ans.**

25. An urn contains 3 white and 6 red balls. Four balls are drawn one by one with replacement from the urn. Find the probability distribution of the number of red balls drawn. Also find mean and variance of the distribution. [6]

Solution : Let X denote the total number of red balls when four balls are drawn one by one with replacement.

P (getting a red ball in one draw) = $\frac{6}{9} = \frac{2}{3}$

P (getting a white ball in one draw) = $\frac{1}{3}$

X	0	1	2	3	4
P(X)	$\left(\frac{1}{3}\right)^4$	$\frac{2}{3}\left(\frac{1}{3}\right)^3 \cdot {}^4C_1$	$\left(\frac{2}{3}\right)^2\left(\frac{1}{3}\right)^2 \cdot {}^4C_2$	$\left(\frac{2}{3}\right)^3\left(\frac{1}{3}\right) \cdot {}^4C_3$	$\left(\frac{2}{3}\right)^4$
	$\frac{1}{81}$	$\frac{8}{81}$	$\frac{24}{81}$	$\frac{32}{81}$	$\frac{16}{81}$

Using the formula for mean, we have

$$\overline{X} = \Sigma P_i X_i$$

Mean

$$(\overline{X}) = \left(0 \times \frac{1}{81}\right) + 1\left(\frac{8}{81}\right) + 2\left(\frac{24}{81}\right) + 3\left(\frac{32}{81}\right) + 4\left(\frac{16}{81}\right)$$

$$= \frac{1}{81}(8 + 48 + 96 + 64)$$

$$= \frac{216}{81}$$

$$= \frac{8}{3}$$

Using the formula for variance, we have

$$\text{Var}(X) = \Sigma P_i X_i^2 - (\Sigma P_i X_i)^2$$

$$\Rightarrow \text{Var}(X) = \left\{\left(0 \times \frac{1}{81}\right) + 1\left(\frac{8}{81}\right) + 4\left(\frac{24}{81}\right) + 9\left(\frac{32}{81}\right) + 16\left(\frac{16}{81}\right)\right\} - \left(\frac{8}{3}\right)^2$$

$$= \frac{648}{81} - \frac{64}{9}$$

$$= \frac{8}{9}$$

Hence, the mean of the disribution is $\frac{8}{3}$ and the variance of the disribution is $\frac{8}{9}$ **Ans.**

26. A manufacturer produces two products A and B. Both the products are processed on two different machines. The available capacity of first machine is 12 hours and that of second machine is 9 hours per day. Each unit of product A requires 3 hours on both machines and each unit of product B requires 2 hours on first machine and 1 hour on second machine. Each unit of product A is sold at ₹ 7 profit and B at a profit of ₹ 4. Find the production level per day for maximum profit graphically. [6]

Solution : Let the numbers of units of products A and B to be produced be x and y, respectively.

Product	Machine	
	I (h)	II (h)
A	3	3
B	2	1

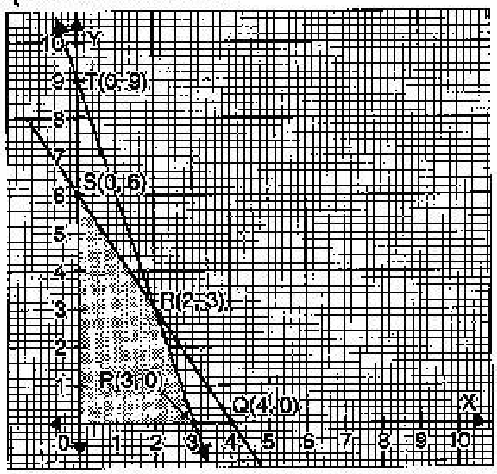
Total profit : $Z = 7x + 4y$

We have to maximize $Z = 7x + 4y$.

Subject to constraints :

$$3x + 2y \le 12$$
$$3x + y \le 9$$
$$\Rightarrow \quad x \ge 0 \text{ and } y \ge 0$$

The given information can be graphically expressed as follows :



Values of $Z = 7x + 4y$ at the corner points are S follows :

Corner Points	$Z = 7x + 4y$
S (0,6)	24
R (2,3)	26 ← Maximum
P (3,0)	21

Therefore, the manufacturer has to produce 2 units of product A and 3 units of product B for the maximum profit of ₹ 26. **Ans.**

●●

All questions are same in Outside Delhi Set II and Set III