Mathematics 2018

Time allowed : 3 hours **Maximum marks : 100**

SECTION – A

1. **Find the value of $\tan^{-1}\sqrt{3}-\cot^{-1}(-\sqrt{3})$** **[1]**

Solution : We have,

$$\tan^{-1}\sqrt{3}-\cot^{-1}(-\sqrt{3})=\tan^{-1}\left(\tan\frac{\pi}{3}\right)-\cot^{-1}\left(\cot\pi-\frac{\pi}{6}\right)$$

$$=\frac{\pi}{3}-\left(\pi-\frac{\pi}{6}\right)$$

$$=\frac{\pi}{3}-\frac{5\pi}{6}=\frac{2\pi-5\pi}{6}$$

$$=\frac{-3\pi}{6}=\frac{-\pi}{2}$$ **Ans.**

2. **If the matrix $A=\begin{bmatrix}0 & a & -3\\ 2 & 0 & -1\\ b & 1 & 0\end{bmatrix}$ is skew-symmetric, find the values of 'a' and 'b'.** **[1]**

Solution : Given, $A=\begin{bmatrix}0 & a & -3\\ 2 & 0 & -1\\ b & 1 & 0\end{bmatrix}$

A is given to be skew-symmetric matrix

$\therefore \quad A^T=-A$

$$\therefore \begin{bmatrix}0 & 2 & b\\ a & 0 & 1\\ -3 & -1 & 0\end{bmatrix}=-\begin{bmatrix}0 & a & -3\\ 2 & 0 & -1\\ b & 1 & 0\end{bmatrix}$$

$$\begin{bmatrix}0 & 2 & b\\ a & 0 & 1\\ -3 & -1 & 0\end{bmatrix}=\begin{bmatrix}0 & -a & 3\\ -2 & 0 & 1\\ -b & -1 & 0\end{bmatrix}$$

On comparing both sides we get

$-a=2$ and $b=3$

$\Rightarrow \quad a=-2$ and $b=3$ **Ans.**

3. **Find the magnitude of each of the two vectors $\vec{a}$ and $\vec{b}$, having the same magnitude such that the angle between them is 60° and their scalar product is $\frac{9}{2}$.** **[1]**

Solution : Let $\vec{a}$ and $\vec{b}$ be two such vectors.

Now, $\vec{a}\cdot\vec{b}=|\vec{a}||\vec{b}|\cos\theta$ ...(i)

It is given that $|\vec{a}|=|\vec{b}|, \theta=60°$ and $\vec{a}\cdot\vec{b}=\frac{9}{2}$

$\therefore$ Putting these values in equation (i)

$$\frac{9}{2}=|\vec{a}||\vec{a}|\cos 60°$$

$$\Rightarrow \frac{9}{2}=|\vec{a}|^2\cdot\frac{1}{2}$$

$$\Rightarrow |\vec{a}|^2=9$$

$$\Rightarrow |\vec{a}|=3$$

$$\therefore |\vec{a}|=|\vec{b}|=3$$ **Ans.**

4. **If $a*b$ denotes the larger of 'a' and 'b' and if $a \; o \; b=(a*b)+3$, then write the value of (5) o (10), where $*$ and o are binary operations.**** **[1]**

SECTION – B

5. **Prove that :**

$3\sin^{-1}x=\sin^{-1}(3x-4x^3),\ x\in\left[-\frac{1}{2},\frac{1}{2}\right]$ **[2]**

Solution : R.H.S. $=\sin^{-1}(3x-4x^3)$

Putting $x=\sin\theta$ in R.H.S, we get

R.H.S. $=\sin^{-1}(3\sin\theta-4\sin^3\theta)$

$=\sin^{-1}(\sin 3\theta)$

Now, $-\frac{1}{2}\le x\le\frac{1}{2}$

$\Rightarrow -\frac{1}{2}\le\sin\theta\le\frac{1}{2}$

$\Rightarrow -\frac{\pi}{6}\le\theta\le\frac{\pi}{6}$

$\Rightarrow -\frac{\pi}{2}\le 3\theta\le\frac{\pi}{2}$

Hence, $\sin^{-1}(\sin 3\theta)=3\theta\left(\text{as } \frac{-\pi}{2}\le 3\theta\le\frac{\pi}{2}\right)$

$\Rightarrow$ R.H.S. $=3\theta$

$=3\sin^{-1}x$

$[\because x=\sin\theta\Rightarrow\theta=\sin^{-1}x]$

= L.H.S. **Hence Proved.**

6. **Given $A=\begin{bmatrix}2 & -3\\ -4 & 7\end{bmatrix}$, compute A^{-1} and show that $2A^{-1}=9I-A$.** **[2]**

Solution : Given, $A=\begin{bmatrix}2 & -3\\ -4 & 7\end{bmatrix}$

$\Rightarrow \quad |A|=14-12=2$

****Answer is not given due to the change in present syllabus**

$$\therefore \quad A^{-1} = \frac{1}{2}\begin{bmatrix} 7 & 3 \\ 4 & 2 \end{bmatrix}$$

Given, $\text{L.H.S.} = 2A^{-1} = 2\left(\frac{1}{2}\right)\begin{bmatrix} 7 & 3 \\ 4 & 2 \end{bmatrix}$

$$\Rightarrow \quad \text{L.H.S.} = \begin{bmatrix} 7 & 3 \\ 4 & 2 \end{bmatrix} \quad \text{...(i)}$$

and $\text{R.H.S.} = 9I - A = 9\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} 2 & -3 \\ -4 & 7 \end{bmatrix}$

$$= \begin{bmatrix} 9 & 0 \\ 0 & 9 \end{bmatrix} - \begin{bmatrix} 2 & -3 \\ -4 & 7 \end{bmatrix}$$

$$\Rightarrow \quad \text{R.H.S.} = \begin{bmatrix} 7 & 3 \\ 4 & 2 \end{bmatrix} \quad \text{...(ii)}$$

From equations (i) and (ii),

L.H.S. = R.H.S. **Hence Proved.**

7. Differentiate $\tan^{-1}\left(\frac{1+\cos x}{\sin x}\right)$ with respect to x. [2]

Solution : Let $y = \tan^{-1}\left(\frac{1+\cos x}{\sin x}\right)$

$$= \tan^{-1}\left(\frac{2\cos^2\frac{x}{2}}{2\sin\frac{x}{2}\cos\frac{x}{2}}\right)$$

$$= \tan^{-1}\left(\cot\frac{x}{2}\right)$$

$$= \tan^{-1}\left[\tan\left(\frac{\pi}{2} - \frac{x}{2}\right)\right]$$

$$= \frac{\pi}{2} - \frac{x}{2}$$

Differentiating w.r.t. x, we get

$$\frac{dy}{dx} = \frac{d}{dx}\left(\frac{\pi}{2} - \frac{x}{2}\right) = -\frac{1}{2} \quad \textbf{Ans.}$$

8. The total cost C(x) associated with the production of x units of an item is given by $C(x) = 0.005x^3 - 0.02x^2 + 30x + 5000$. Find the marginal cost when 3 units are produced, where by marginal cost we mean the instantaneous rate of change of total cost at any level of output. [2]

Solution : Cost function is given as

$$C(x) = 0.005x^3 - 0.02x^2 + 30x + 5000$$

$$\text{Marginal cost (MC)} = \frac{d}{dx}(C(x))$$

$$= 0.005(3x^2) - 0.02(2x) + 30$$

$$= 0.015x^2 - 0.04x + 30$$

When $x = 3$, $MC = 0.015(3)^2 - 0.04(3) + 30$

$$= 0.135 - 0.12 + 30$$

$$= 30.015 \quad \textbf{Ans.}$$

9. Evaluate : $\int \frac{\cos 2x + 2\sin^2 x}{\cos^2 x}dx$

Solution : We have,

$$\int \frac{\cos 2x + 2\sin^2 x}{\cos^2 x}dx = \int \frac{\cos^2 x - \sin^2 x + 2\sin^2 x}{\cos^2 x}dx$$

$$(\because \cos 2x = \cos^2 x - \sin^2 x)$$

$$= \int \frac{\cos^2 x + \sin^2 x}{\cos^2 x}dx$$

$$= \int \frac{1}{\cos^2 x}dx$$

$$= \int \sec^2 x\,dx$$

$$= \tan x + C \quad \textbf{Ans.}$$

10. Find the differential equation representing the family of curves $y = ae^{bx+5}$, where a and b are arbitrary constants. [2]

Solution : Given curve is

$$y = ae^{bx+5} \quad \text{...(i)}$$

Differentiating (i) w.r.t. x, we get

$$\frac{dy}{dx} = ae^{bx+5} \cdot \frac{d}{dx}(bx+5)$$

$$\Rightarrow \quad \frac{dy}{dx} = ae^{bx+5} \cdot b \quad \text{...(ii)}$$

$$\Rightarrow \quad \frac{dy}{dx} = y \cdot b \quad \text{[From (i)]}$$

Differentiating again w.r.t. x, we get

$$\frac{d^2y}{dx^2} = \frac{dy}{dx} \cdot b$$

$$\Rightarrow \quad \frac{d^2y}{dx^2} = \frac{dy}{dx}\left(\frac{dy}{dx} \cdot \frac{1}{y}\right) \quad \left[\because b = \frac{1}{y}\frac{dy}{dx}\right]$$

$$\Rightarrow \quad y\frac{d^2y}{dx^2} = \left(\frac{dy}{dx}\right)^2 \quad \textbf{Ans.}$$

11. If θ is the angle between two vectors $\hat{i} - 2\hat{j} + 3\hat{k}$ and $3\hat{i} - 2\hat{j} + \hat{k}$, find $\sin\theta$. [2]

Solution : Let $\vec{a} = \hat{i} - 2\hat{j} + 3\hat{k}$

$$\vec{b} = 3\hat{i} - 2\hat{j} + \hat{k}$$

We know, $\vec{a} \cdot \vec{b} = |\vec{a}||\vec{b}|\cos\theta$

$$\Rightarrow \quad \cos\theta = \frac{\vec{a} \cdot \vec{b}}{|\vec{a}||\vec{b}|}$$

$$= \frac{(\hat{i} - 2\hat{j} + 3\hat{k}) \cdot (3\hat{i} - 2\hat{j} + \hat{k})}{|\hat{i} - 2\hat{j} + 3\hat{k}|\,|3\hat{i} - 2\hat{j} + \hat{k}|}$$

$$= \frac{1(3)+(-2)(-2)+3(1)}{\sqrt{1+4+9}\sqrt{9+4+1}}$$

$$= \frac{3+4+3}{14} = \frac{10}{14} = \frac{5}{7}$$

Now, $\sin\theta = \sqrt{1-\cos^2\theta}$

$$= \sqrt{1-\left(\frac{5}{7}\right)^2} = \sqrt{1-\frac{25}{49}}$$

$$= \sqrt{\frac{49-25}{49}}$$

$\Rightarrow$ $\sin\theta = \sqrt{\frac{24}{49}} = \frac{2\sqrt{6}}{7}$ **Ans.**

12. A black and a red die are rolled together. Find the conditional probability of obtaining the sum 8, given that the red die resulted in a number less than 4. **[2]**

Solution : The sample space has 36 Outcomes. Let A be event that the sum of observations is 8.

$\therefore$ $A = \{(2,6),(3,5),(5,3),(4,4),(6,2)\}$

$\Rightarrow$ $n(A) = 5$

$\Rightarrow$ $P(A) = \frac{5}{36}$

Let B be event that observation on red die is less than 4.

$\therefore$ $B = \{(1,1),(2,1),(3,1),(4,1),(5,1),(6,1),(1,2),(2,2),(3,2),(4,2),(5,2),(6,2),(1,3),(2,3),(3,3),(4,3),(5,3),(6,3)\}$

$\therefore$ $n(B) = 18$

$\Rightarrow$ $P(B) = \frac{18}{36} = \frac{1}{2}$

Clearly $A \cap B = \{(5,3),(6,2)\}$

$\therefore$ $n(A \cap B) = 2$

$\therefore$ $P(A \cap B) = \frac{2}{36} = \frac{1}{18}$

$$P\left(\frac{A}{B}\right) = \frac{P(A\cap B)}{P(B)} = \frac{\frac{1}{18}}{\frac{1}{2}}$$

$= \frac{2}{18} = \frac{1}{9}$ **Ans.**

SECTION–C

13. Using properties of determinants, prove that :

$$\begin{vmatrix} 1 & 1 & 1+3x \\ 1+3y & 1 & 1 \\ 1 & 1+3z & 1 \end{vmatrix} = 9(3xyz + xy + yz + zx)$$ **[4]**

Solution :

$$\text{L.H.S.} = \begin{vmatrix} 1 & 1 & 1+3x \\ 1+3y & 1 & 1 \\ 1 & 1+3z & 1 \end{vmatrix}$$

Taking x, y and z common from R_1, R_2 and R_3 respectively, we get

$$= xyz\begin{vmatrix} \frac{1}{x} & \frac{1}{x} & \frac{1}{x}+3 \\ \frac{1}{y}+3 & \frac{1}{y} & \frac{1}{y} \\ \frac{1}{z} & \frac{1}{z}+3 & \frac{1}{z} \end{vmatrix}$$

On applying $R_1 \to R_1 + R_2 + R_3$, we get

$$= xyz\begin{vmatrix} \frac{1}{x}+\frac{1}{y}+\frac{1}{z}+3 & \frac{1}{x}+\frac{1}{y}+\frac{1}{z}+3 & \frac{1}{x}+\frac{1}{y}+\frac{1}{z}+3 \\ \frac{1}{y}+3 & \frac{1}{y} & \frac{1}{y} \\ \frac{1}{z} & \frac{1}{z}+3 & \frac{1}{z} \end{vmatrix}$$

Taking common $\left(\frac{1}{x}+\frac{1}{y}+\frac{1}{z}+3\right)$ from R_1, we get

$$= xyz\left(\frac{1}{x}+\frac{1}{y}+\frac{1}{z}+3\right)\begin{vmatrix} 1 & 1 & 1 \\ \frac{1}{y}+3 & \frac{1}{y} & \frac{1}{y} \\ \frac{1}{z} & \frac{1}{z}+3 & \frac{1}{z} \end{vmatrix}$$

On applying $C_2 \to C_2 - C_1$ and $C_3 \to C_3 - C_1$, we get

$$= xyz\left(\frac{1}{x}+\frac{1}{y}+\frac{1}{z}+3\right)\begin{vmatrix} 1 & 0 & 0 \\ \frac{1}{y}+3 & -3 & -3 \\ \frac{1}{z} & & \end{vmatrix}$$

On expanding along R_1, we get

$$= xyz\left(\frac{1}{x}+\frac{1}{y}+\frac{1}{z}+3\right)1\,(0+9)$$

$$= xy\left(\frac{yz+xz+xy+3xyz}{xyz}\right)(9)$$

$$= 9\,(3xyz + xy + yz + zx)$$

= R.H.S. **Hence Proved.**

14. If $(x^2 + y^2)^2 = xy$, find $\frac{dy}{dx}$. **[4]**

Solution : We have,

$(x^2 + y^2)^2 = xy$ ...(i)

Differentiating (i) w.r.t. x, we get

$$2(x^2+y^2)\left(2x+2y.\frac{dy}{dx}\right) = x\frac{dy}{dx}+y$$

$$4x\,(x^2+y^2)+4y\frac{dy}{dx}(x^2+y^2) = x\frac{dy}{dx}+y$$

$$\Rightarrow \quad \frac{dy}{dx}\,[4y\,(x^2+y^2)-x] = y-4x\,(x^2+y^2)$$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{y-4x(x^2+y^2)}{4y(x^2+y^2)-x}$$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{y-4x^3-4xy^2}{4x^2y+4y^3-x}$$

Ans.

OR

If $x = a\,(2\theta - \sin 2\theta)$ and $y = a\,(1-\cos 2\theta)$, find $\frac{dy}{dx}$ when $\theta = \frac{\pi}{3}$.

Solution : Given, $x = a\,(2\theta - \sin 2\theta)$

and $\quad y = a\,(1-\cos 2\theta)$

Differentiating x and y w.r.t. θ, we get

$$\frac{dx}{d\theta} = a(2-\cos 2\theta\cdot 2)$$

$$= 2a(1-\cos 2\theta)$$

and $$\frac{dy}{d\theta} = a\,(\sin 2\theta\cdot 2)$$

$$= 2a\sin 2\theta$$

$$\therefore \quad \frac{dy}{dx} = \frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}} = \frac{2a\sin 2\theta}{2a(1-\cos 2\theta)}$$

$$= \frac{\sin 2\theta}{1-\cos 2\theta}$$

$$= \frac{2\sin\theta\cos\theta}{2\sin^2\theta}$$

$$= \cot\theta$$

$$\therefore \quad \left[\frac{dy}{dx}\right]_{\theta=\frac{\pi}{3}} = \cot\frac{\pi}{3} = \frac{1}{\sqrt{3}}$$

Ans.

15. If $y = \sin(\sin x)$, prove that :

$$\frac{d^2y}{dx^2}+\tan x\frac{dy}{dx}+y\cos^2 x = 0$$

[4]

Solution : Given, $y = \sin(\sin x)$

Differentiating y w.r.t. x, we get

$$\frac{dy}{dx} = \cos(\sin x)\cdot\cos x$$

Differentiating again w.r.t. x, we get

$$\frac{d^2y}{dx^2} = \cos(\sin x)(-\sin x)+\cos x\,[-\sin(\sin x)\cdot\cos x]$$

$$= -\sin x.\cos(\sin x)-\cos^2 x\cdot\sin(\sin x)$$

Now, $\quad$ L.H.S. $= \frac{d^2y}{dx^2}+\tan x\frac{dy}{dx}+y\cos^2 x$

$$= -\sin x\cdot\cos(\sin x)-\cos^2 x\cdot\sin(\sin x) + \tan x\,[\cos(\sin x)\cdot\cos x] + \cos^2 x\cdot\sin(\sin x)$$

$$= -\sin x\cos(\sin x)+\frac{\sin x}{\cos x}[\cos(\sin x)\cdot\cos x]$$

$$= -\sin x\cdot\cos(\sin x)+\sin x\cdot\cos(\sin x)$$

$= 0 =$ R.H.S. $\quad$ **Hence Proved.**

16. Find the equations of the tangent and the normal to the curve $16x^2+9y^2 = 145$ at the point (x_1, y_1), where $x_1 = 2$ and $y_1 > 0$. [4]

Solution : Given curve is

$$16x^2+9y^2 = 145 \quad \text{....(i)}$$

Since (x_1, y_1) lies on equation (i),

$$\therefore \quad 16x_1^2+9y_1^2 = 145$$

$$\Rightarrow \quad 16\,(2)^2+9y_1^2 = 145 \quad [\because x_1 = 2 \text{ (given)}]$$

$$\Rightarrow \quad 9y_1^2 = 145-64 = 81$$

$$\Rightarrow \quad y_1 = 3 \quad [\because y_1 > 0 \text{ (given)}]$$

$\therefore$ Point of contact is (2, 3).

Differentiating equation (i) w.r.t. x, which will give us the slope of the tangent.

$$32x+18y\frac{dy}{dx} = 0$$

$$\Rightarrow \quad \frac{dy}{dx} = -\frac{32x}{18y}$$

Slope of tangent at (2, 3)

$$= \left[\frac{dy}{dx}\right]_{(2,3)}$$

$$= -\frac{32(2)}{18(3)} = -\frac{64}{54}$$

$$= -\frac{32}{27}$$

$\therefore$ Equation of tangent is

$$y-3 = m(x-2)$$

$$\Rightarrow \quad y-3 = -\frac{32}{27}(x-2)$$

$$\Rightarrow \quad 27y-81 = -32x+64$$

$$\Rightarrow \quad 32x+27y = 145$$

The slope of the normal $= \frac{-1}{\text{Slope of tangent}}$

$$= \frac{27}{32}$$

$\therefore$ Equation of normal is

$$\Rightarrow \quad y-3 = \frac{27}{32}(x-2)$$

$$\Rightarrow \quad 32y-96 = 27x-54$$

$$\Rightarrow \quad 27x-32y = -42$$ **Ans.**

OR

Find the intervals in which the function

$$f(x) = \frac{x^4}{4} - x^3 - 5x^2 + 24x + 12 \text{ is}$$

(a) strictly increasing,
(b) strictly decreasing.

Solution : Given function is

$$f(x) = \frac{x^4}{4} - x^3 - 5x^2 + 24x + 12$$

$$\Rightarrow \quad f'(x) = \frac{4x^3}{4} - 3x^2 - 10x + 24$$

$$\Rightarrow \quad f'(x) = x^3 - 3x^2 - 10x + 24$$

For critical points, put $f'(x) = 0$

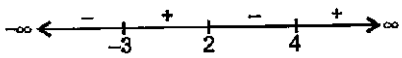
$$\therefore \quad x^3 - 3x^2 - 10x + 24 = 0$$

$$\Rightarrow \quad (x-2)(x^2 - x - 12) = 0$$

$$\Rightarrow \quad (x-2)(x-4)(x+3) = 0$$

$$\Rightarrow \quad x = 2, 4, -3$$

Therefore, we have the intervals $(-\infty, -3)$, $(-3, 2)$, $(2, 4)$ and $(4, \infty)$



Since $f'(x) > 0$ in $(-3, 2) \cup (4, \infty)$.

$\therefore$ $f(x)$ is strictly increasing in interval $(-3, 2) \cup (4, \infty)$

and $\quad f'(x) < 0$ in $(-\infty, -3) \cup (2, 4)$

$\therefore$ $f(x)$ is strictly decreasing in $(-\infty, -3) \cup (2, 4)$.

Ans.

17. **An open tank with a square base and vertical sides is to be constructed from a metal sheet so as to hold a given quantity of water. Show that the cost of material will be least when depth of the tank is half of its width. If the cost is to be borne by nearby settled lower income families, for whom water will be provided, what kind of value is hidden in this question ? [4]**

Solution : Let the length, breadth and height of the open tank be x, x and y units respectively.

Then, Volume (V) $= x^2 y$ ...(i)

Total surface area (S) $= x^2 + 4xy$ ...(ii)

$$S = x^2 + 4x\frac{V}{x^2} \quad \text{[using (i)]}$$

$$\Rightarrow \quad S = x^2 + \frac{4V}{x}$$

$$\Rightarrow \quad \frac{dS}{dx} = 2x - \frac{4V}{x^2}$$

For critical points, put

$$\frac{dS}{dx} = 0$$

$$\Rightarrow \quad 2x - \frac{4V}{x^2} = 0$$

$$\Rightarrow \quad 2x^3 = 4V$$

$$\Rightarrow \quad 2x^3 = 4x^2 y \quad \text{[using (i)]}$$

$$\Rightarrow \quad x = 2y \quad \text{...(iii)}$$

Now, $$\frac{d^2S}{dx^2} = 2 + \frac{8V}{x^3}$$

$$= 2 + \frac{8V}{8y^3} \quad \text{[using (iii)]}$$

$$= 2 + \frac{V}{y^3} > 0$$

Area is minimum, thus cost is minimum when $x = 2y$.

i.e., depth of tank is half of the width.

Value : Any relevant value. **Ans.**

18. **Find** $\displaystyle\int \frac{2\cos x}{(1 \quad \sin x)(1+\sin^2 x)}dx$ **[4]**

Solution : Let $\displaystyle I = \int \frac{2\cos x}{(1-\sin x)(1+\sin^2 x)}dx$

Put $\quad \sin x = t$

$\Rightarrow \quad \cos x \, dx = dt$

$$\therefore \quad I = \int \frac{2dt}{(1-t)(1+t^2)}$$

Let $$\frac{2}{(1-t)(1+t^2)} = \frac{A}{1-t} + \frac{Bt+C}{1+t^2}$$

$$2 = A(1+t^2) + (Bt+C)(1-t)$$

$$\Rightarrow \quad 2 = (A-B)t^2 + (B-C)t + A + C$$

Put $t = 1$, we get

$$2 = 2A$$

$$\Rightarrow \quad A = 1$$

Comparing coefficients of t^2 and t on both sides,

$$A - B = 0$$

$$\Rightarrow \quad B = A$$

$$\Rightarrow \quad B = 1$$

Also, $\quad B - C = 0$

$$\Rightarrow \quad B = C = 1$$

$$\therefore \quad I = \int \left(\frac{1}{1-t} + \frac{t+1}{t^2+1}\right)dt$$

$$= \frac{\log(1-t)}{-1} + \frac{1}{2}\int \frac{2t}{t^2+1}dt + \int \frac{1}{t^2+1}dt$$

$$= -\log|1-t| + \frac{1}{2}\log|t^2+1| + \tan^{-1} t + C$$

$$= -\log(1-\sin x) + \frac{1}{2}\log(\sin^2 x + 1) + \tan^{-1}(\sin x) + C$$

Ans.

19. Find the particular solution of the differential equation $e^x \tan y\, dx + (2 - e^x) \sec^2 y\, dy = 0$, given that $y = \frac{\pi}{4}$ when $x = 0$. **[4]**

Solution : Given differential equation is,

$e^x \tan y\, dx + (2 - e^x) \sec^2 y\, dy = 0$

$\Rightarrow \quad e^x \tan y\, dx = (e^x - 2) \sec^2 y\, dy$

$$\Rightarrow \quad \frac{e^x}{e^x - 2} dx = \frac{\sec^2 y}{\tan y} dy$$

Integrating both sides, we get

$$\int \frac{e^x}{e^x - 2} dx = \int \frac{\sec^2 y}{\tan y} dy$$

$\Rightarrow \quad \log |e^x - 2| = \log |\tan y| + \log C$

$\Rightarrow \quad \log |e^x - 2| = \log |C \tan y|$

$\Rightarrow \quad e^x - 2 = C \tan y$ ...(i)

Put $y = \frac{\pi}{4}$ when $x = 0$

$$e^0 - 2 = C\left(\tan\frac{\pi}{4}\right)$$

$\Rightarrow \quad -1 = C$

$\Rightarrow \quad C = -1$

$\therefore$ From equation (i),

$$e^x - 2 = -\tan y \Rightarrow y = \tan^{-1}(2 - e^x)$$

which is the required solution. **Ans.**

OR

Find the particular solution of the differential equation $\frac{dy}{dx} + 2y \tan x = \sin x$, given than $y = 0$ when $x = \frac{\pi}{3}$.

Solution : Given differential equation is,

$$\frac{dy}{dx} + (2\tan x)y = \sin x \quad \text{....(i)}$$

which is linear differential equation of the form

$$\frac{dy}{dx} + Py = Q$$

where $P = 2 \tan x$, $Q = \sin x$

Integrating factor,

$$\text{I.F.} = e^{\int P.dx} = e^{\int 2\tan x.dx} = e^{2\log \sec x} = \sec^2 x$$

Solution of equation (i) is given by

$y \cdot \text{I.F.} = \int Q \cdot \text{I.F.}\, dx + C$

$\Rightarrow \quad y \sec^2 x = \int \sin x \cdot \sec^2 x \cdot dx + C$

$\Rightarrow \quad y \sec^2 x = \int \sec x \cdot \tan x \cdot dx + C$

$\Rightarrow \quad y \sec^2 x = \sec x + C$

Put $y = 0$ when $x = \frac{\pi}{3}$

$$0.\sec^2\frac{\pi}{3} = \sec\frac{\pi}{3} + C$$

$\Rightarrow \quad 0 = 2 + C$

$\Rightarrow \quad C = -2$

Hence, particular solution is

$$y \sec^2 x = \sec x - 2$$

or $\quad y = \cos x - 2\cos^2 x$ **Ans.**

20. Let $\vec{a} = 4\hat{i} + 5\hat{j} - \hat{k}$, $\vec{b} = \hat{i} - 4\hat{j} + 5\hat{k}$ and $\vec{c} = 3\hat{i} + \hat{j} - \hat{k}$. Find a vector $\vec{d}$ which is perpendicular to both $\vec{c}$ & $\vec{b}$ and $\vec{d} \cdot \vec{a} = 21$. **[4]**

Solution : Given,

$\vec{a} = 4\hat{i} + 5\hat{j} - \hat{k}$, $\vec{b} = \hat{i} - 4\hat{j} + 5\hat{k}$, $\vec{c}\ 3\hat{i} + \hat{j} - \hat{k}$

Let, $\quad \vec{d} = x\hat{i} + y\hat{j} + z\hat{k}$

$\because \quad \vec{d}$ is perpendicular to $\vec{c}$ and $\vec{b}$

$\therefore \quad \vec{d}.\vec{b} = 0$

$x - 4y + 5z = 0$ ...(i)

and $\quad \vec{d} \cdot \vec{c} = 0$

$3x + y - z = 0$ ...(ii)

Also $\quad \vec{d}.\vec{a} = 21$ (given)

$\Rightarrow \quad 4x + 5y - z = 21$ ...(iii)

On subtracting equation (ii) from equation (iii), we get

$x + 4y = 21$ ...(iv)

On multiplying equation (iii) by 5 and adding equation (i), we get

$21x + 21y = 105$

$\Rightarrow \quad x + y = 5$ ....(v)

On subtracting equation (v) from equation (iv), we get

$$3y = 16 \Rightarrow y = \frac{16}{3}$$

From equation (v),

$$x = 5 - \frac{16}{3} = -\frac{1}{3}$$

From equation (ii),

$$z = 3x + y$$

$$= 3\left(-\frac{1}{3}\right) + \frac{16}{3} = \frac{13}{3}$$

$$\therefore \quad \vec{d} = \frac{-1}{3}\hat{i} + \frac{16}{3}\hat{j} + \frac{13}{3}\hat{k}$$ **Ans.**

21. Find the shortest distance between the lines $\vec{r} = (4\hat{i} - \hat{j}) + \lambda(\hat{i} + 2\hat{j} - 3\hat{k})$ and $\vec{r} = (\hat{i} - \hat{j} + 2\hat{k}) + \mu(2\hat{i} + 4\hat{j} - 5\hat{k})$, **[4]**

Solution : Given lines are

$\vec{r} = (4\hat{i} - \hat{j}) + \lambda(\hat{i} + 2\hat{j} - 3\hat{k})$ ...(i)

and $\vec{r} = (\hat{i} - \hat{j} + 2\hat{k}) + \mu(2\hat{i} + 4\hat{j} - 5\hat{k})$, ...(ii)

Comparing equation (i) with $\vec{r} = \vec{a}_1 + \lambda\vec{b}_1$ and equation (ii) with $\vec{r} = \vec{a}_2 + \mu\vec{b}_2$, we get

$$\vec{a}_1 = 4\hat{i} - \hat{j}$$

$$\vec{a_2} = \hat{i} - \hat{j} + 2\hat{k}$$
$$\vec{b_1} = \hat{i} - 2\hat{j} + 3\hat{k}$$
$$\vec{b_2} = 2\hat{i} + 4\hat{j} - 5\hat{k}$$

Shortest distance between equation (i) and (ii) is given by

$$\text{S.D.} = \left|\frac{(\vec{a_2} - \vec{a_1}).(\vec{b_1} \times \vec{b_2})}{|\vec{b_1} \times \vec{b_2}|}\right|$$

$$\vec{b_1} \times \vec{b_2} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 2 & -3 \\ 2 & 4 & -5 \end{vmatrix}$$

$$\Rightarrow \quad \vec{b_1} \times \vec{b_2} = \hat{i}(-10+12) - \hat{j}(-5+6) + \hat{k}(4-4)$$
$$= 2\hat{i} - \hat{j}.$$

$$\Rightarrow \quad |\vec{b_1} \times \vec{b_2}| = \sqrt{4+1} = \sqrt{5}$$

$$\Rightarrow \quad \text{S.D.} = \left|\frac{(-3\hat{i} + 0\hat{j} + 2\hat{k})\cdot(2\hat{i} - \hat{j})}{\sqrt{4+1}}\right| \left|-\frac{6}{\sqrt{5}}\right|$$

$$= \frac{6}{\sqrt{5}} \text{ or } \frac{6\sqrt{5}}{5} \text{ units}$$ **Ans.**

22. Suppose a girl throws a die. If she gets 1 or 2, she tosses a coin three times and notes the number of tails. If she gets 3, 4, 5 or 6, she tosses a coin once and notes whether a 'head' or 'tail' is obtained. If she obtained exactly one 'tail', what is the probability that she threw 3, 4, 5 or 6 with the dice? [4]

Solution : Let E_1 be the event that girl gets 1 or 2 on the roll and E_2 be the event that girl gets 3, 4, 5, or 6 on the roll of a die.

$$\therefore \quad P(E_1) = \frac{2}{6} = \frac{1}{3}$$

$$P(E_2) = \frac{4}{6} = \frac{2}{3}$$

Let A be the event that she gets exactly one tail. If she tossed coin 3 times and gets exactly one tail then possible outcomes are HTH, HHT, THH

$$\therefore \quad P(A/E_1) = \frac{3}{8}$$

If she tossed coin only once and exactly one tail shows

Then, $P(A/E_2) = \frac{1}{2}$

$$\therefore \; P(E_2/A) = \frac{P(E_2)\cdot P(A/E_2)}{P(E_1)\cdot P(A/E_1) + P(E_2)\cdot P(A/E_2)}$$
$$= \frac{\frac{2}{3} \times \frac{1}{2}}{\frac{1}{3} \times \frac{3}{8} + \frac{2}{3} \times \frac{1}{2}} = \frac{\frac{1}{3}}{\frac{1}{8} + \frac{1}{3}} = \frac{\frac{1}{3}}{\frac{11}{24}}$$

$$= \frac{8}{11}$$ **Ans.**

23. Two numbers are selected at random (without replacement) from the first five positive integers. Let X denote the larger of the two numbers obtained. Find the mean and variance of X. [4]

Solution : First five positive integers are 1, 2, 3, 4, 5. We select two positive numbers in $5 \times 4 = 20$ ways. Out of these, two numbers are selected at random. Let X denote larger of the two selected numbers. Then, X can have values 2, 3, 4 or 5.

$$P(X=2) = P(\text{larger no. is } 2) = \{(1, 2) \text{ and } (2, 1)\}$$
$$= \frac{2}{20}$$
$$P(X=3) = \frac{4}{20}$$
$$P(X=4) = \frac{6}{20}$$
$$P(X=5) = \frac{8}{20}$$

Thus, the probability distribution of X is

X	2	3	4	5
P(X)	$\frac{2}{20}$	$\frac{2}{20}$	$\frac{6}{20}$	$\frac{8}{20}$

$$\therefore \quad \text{Mean} = E(X) = \sum_{i=1}^{n} x_i \, p(x_i)$$
$$= 2 \times \frac{2}{20} + 3 \times \frac{4}{20} + 4 \times \frac{6}{20} + 5 \times \frac{8}{20}$$
$$= \frac{4+12+24+40}{20} = \frac{80}{20} = 4$$

$$E(X^2) = \sum_{i=1}^{n} x_i^2 \, p(x_i)$$
$$= 2^2 \times \frac{2}{20} + 3^2 \times \frac{4}{20} + 4^2 \times \frac{6}{20} + 5^2 \times \frac{8}{20}$$
$$= \frac{8}{20} + \frac{36}{20} + \frac{96}{20} + \frac{200}{20}$$
$$= \frac{8+36+96+200}{20} = \frac{340}{20} = \frac{34}{2} = 17$$

$$\text{Variance} = E(X^2) - [E(X)]^2$$
$$= 17 - (4)^2 = 1$$

Therefore, mean and variance are 4 and 1 respectively. **Ans.**

SECTION - D

24. Let $A = \{x \in Z : 0 \le x \le 12\}$. Show that $R = \{(a, b) : a, b \in A, |a - b|$ is divisible by $4\}$ is an equivalence relation. Find the set of all elements related to 1. Also write the equivalence class [2]. [6]

Solution : Given, R = {(*a*, *b*) : *a*, *b* ∈ *A*, | *a* − *b* | is divisible by 4}

Reflexivity : For any $a \in A$

$$|a - a| = 0, \text{ which is divisible by 4}$$

$$(a, a) \in R$$

So, *R* is reflexive.

Symmetry : Let $(a, b) \in R$

$\Rightarrow |a - b|$ is divisible by 4

$\Rightarrow |b - a|$ is divisible by 4

$$[\because |a - b| = |b - a|]$$

$\Rightarrow (b, a) \in R$

So, R is symmetric.

Transitive : Let $(a, b) \in R$ and $(b, c) \in R$

$\Rightarrow |a - b|$ is divisible by 4

$\Rightarrow |a - b| = 4k$

$\therefore a - b = \pm 4k, k \in Z$ ...(i)

Also, $|b - c|$ is divisible by 4

$\Rightarrow |b - c| = 4m$

$\therefore b - c = \pm 4m, m \in Z$ ...(ii)

Adding equations (i) and (ii)

$a - b + b - c = \pm 4(k + m)$

$\Rightarrow a - c = \pm 4(k + m)$

$|a - c|$ is divisible by 4,

$\Rightarrow (a, c) \in R$

So, R is transitive.

⇒ R is reflexive, symmetric and transitive.

∴ R is an equivalence relation.

Let *x* be an element of R such that $(x, 1) \in R$

Then $|x - 1|$ is divisible by 4

$x - 1 = 0, 4, 8, 12, ...$

$\Rightarrow x = 1, 5, 9$ $(\because x \le 12)$

∴ Set of all elements of A which are related to 1 are {1, 5, 9}.

Equivalence class of 2 *i.e.*

$[2] = \{(a, 2) : a \in A, |a - 2| \text{ is divisible by 4}\}$

$\Rightarrow |a - 2| = 4k$ (k is whole number, $k \le 3$)

$\Rightarrow a = 2, 6, 10$

Therefore, equivalence class [2] is {2, 6, 10}. **Ans.**

OR

Show that the function $f : R \to R$ defined by $f(x) = \frac{x}{x^2+1}, \forall x \in R$ is neither one-one nor onto. Also, if $g : R \to R$ is defined as $g(x) = 2x - 1$, find $f o g(x)$.

Solution : Given, $f(x) = \frac{x}{x^2+1}, \forall x \in R$

For one−one, $f(x) = f(y)$

$$\frac{x}{x^2+1} = \frac{y}{y^2+1}$$

$\Rightarrow xy^2 + x = yx^2 + y$

$\Rightarrow xy^2 - yx^2 = y - x$

$\Rightarrow xy(y - x) = y - x$

$\Rightarrow xy = 1$

$\Rightarrow x = \frac{1}{y}$

Since $x \ne y$, therefore, $f(x)$ is not one-one.

For onto, $f(x) = y$

$\Rightarrow \frac{x}{x^2+1} = y$

$\Rightarrow x = yx^2 + y$

$\Rightarrow x^2y + y - x = 0$

x cannot be expressed in terms of *y*

⇒ *f*(*x*) is not onto.

As $g(x) = 2x - 1$

$\therefore f o g(x) = f[g(x)] = f(2x - 1)$

$$= \frac{2x-1}{(2x-1)^2+1} = \frac{2x-1}{4x^2-4x+2} \text{ Ans.}$$

25. If $A = \begin{bmatrix} 2 & -3 & 5 \\ 3 & 2 & -4 \\ 1 & 1 & -2 \end{bmatrix}$, find A^{-1}. Use it to solve the system of equations : **[6]**

$$2x - 3y + 5z = 11$$
$$3x + 2y - 4z = -5$$
$$x + y - 2z = -3$$

Solution : $|A| = \begin{vmatrix} 2 & -3 & 5 \\ 3 & 2 & -4 \\ 1 & 1 & -2 \end{vmatrix}$

$= 2(-4+4) + 3(-6+4) + 5(3-2)$

$|A| = 0 - 6 + 5 = -1 \ne 0$

∴ A^{-1} exists.

Let A_{ij} be the cofactors of elements a_{ij} in $A = [a_{ij}]$, then

$A_{11} = (-1)^{1+1} \begin{vmatrix} 2 & -4 \\ 1 & -2 \end{vmatrix} = -4 + 4 = 0$

$A_{12} = (-1)^{1+2} \begin{vmatrix} 3 & -4 \\ 1 & -2 \end{vmatrix} = -(-6 + 4) = 2$

$A_{13} = (-1)^{1+3} \begin{vmatrix} 3 & 2 \\ 1 & 1 \end{vmatrix} = 3 - 2 = 1$

$A_{21} = (-1)^{2+1} \begin{vmatrix} -3 & 5 \\ 1 & -2 \end{vmatrix} = -(6 - 5) = -1$

$A_{22} = (-1)^{2+2} \begin{vmatrix} 2 & 5 \\ 1 & -2 \end{vmatrix} = -4 - 5 = -9$

$$A_{23} = (-1)^{2+3} \begin{vmatrix} 2 & -3 \\ 1 & 1 \end{vmatrix} = -(2+3) = -5$$

$$A_{31} = (-1)^{3+1} \begin{vmatrix} -3 & 5 \\ 2 & -4 \end{vmatrix} = (12-10) = 2$$

$$A_{32} = (-1)^{3+2} \begin{vmatrix} 2 & 5 \\ 3 & -4 \end{vmatrix} = -(-8-15) = 23$$

$$A_{33} = (-1)^{3+3} \begin{vmatrix} 2 & -3 \\ 3 & 2 \end{vmatrix} = 4+9 = 13$$

$$\therefore \quad A_{ij} = \begin{bmatrix} 0 & 2 & 1 \\ -1 & -9 & -5 \\ 2 & 23 & 13 \end{bmatrix}$$

$$\text{adj}(A) = [A_{ij}]^T = \begin{bmatrix} 0 & -1 & 2 \\ 2 & -9 & 23 \\ 1 & -5 & 13 \end{bmatrix}$$

$$A^{-1} = \frac{\text{adj}(A)}{|A|} = \frac{1}{(-1)} \begin{bmatrix} 0 & -1 & 2 \\ 2 & -9 & 23 \\ 1 & -5 & 13 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 1 & -2 \\ -2 & 9 & -23 \\ -1 & 5 & -13 \end{bmatrix}$$

Now, given system of equations can be written as

$$\begin{bmatrix} 2 & -3 & -5 \\ 3 & 2 & -4 \\ 1 & 1 & -2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 11 \\ -5 \\ -3 \end{bmatrix}$$

$$\Rightarrow \quad AX = B$$

$$\therefore \quad X = A^{-1}B$$

$$X = \begin{bmatrix} 0 & 1 & -2 \\ -2 & 9 & -23 \\ -1 & 5 & -13 \end{bmatrix} \begin{bmatrix} 11 \\ -5 \\ -3 \end{bmatrix}$$

$$\Rightarrow \quad \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0-5+6 \\ -22-45+69 \\ -11-25+39 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$

$\therefore \quad x = 1, y = 2, z = 3$ **Ans.**

OR

Using elementary row transformations, find the inverse of the matrix:

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 5 & 7 \\ -2 & -4 & -5 \end{bmatrix}.$$

$$\begin{bmatrix} 1 & 2 & 3 \\ 2 & 5 & 7 \\ -2 & -4 & -5 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} A$$

On applying, $R_2 \to R_2 - 2R_1$ and $R_3 \to R_3 + 2R_1$, we get

$$\begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 2 & 0 & 1 \end{bmatrix} A$$

On applying $R_1 \to R_1 - 3R_3$ and $R_2 \to R_2 - R_3$, we get

$$\begin{bmatrix} 1 & 2 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} -5 & 0 & -3 \\ -4 & 1 & -1 \\ 2 & 0 & 1 \end{bmatrix} A$$

On applying $R_1 \to R_1 - 2R_2$, we get

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 3 & -2 & -1 \\ -4 & 1 & -1 \\ 2 & 0 & 1 \end{bmatrix} A$$

$$\therefore \quad A^{-1} = \begin{bmatrix} 3 & -2 & -1 \\ -4 & 1 & -1 \\ 2 & 0 & 1 \end{bmatrix}$$ **Ans.**

26. Using integration, find the area of the region in the first quadrant enclosed by the X-axis, the line $y = x$ and the circle $x^2 + y^2 = 32$. **[6]**

Solution : Given curve is

$$x^2 + y^2 = 32 \quad \text{... (i)}$$

$$x^2 + y^2 = \left(\sqrt{32}\right)^2 = \left(4\sqrt{2}\right)^2$$

It is a circle with centre (0, 0) and radius $4\sqrt{2}$.

Given line is $y = x$ ....(ii)

Solving equations (i) and (ii) for points of intersections,

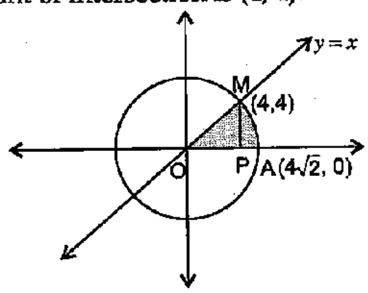
$$x^2 + x^2 = 32 \quad \text{[using (ii) in (i)]}$$

$$\Rightarrow \quad x^2 = 16$$

$$\Rightarrow \quad x = \pm 4$$

$$\Rightarrow \quad x = 4 \quad \text{(first quadrant)}$$

$\therefore$ Point of intersection is (4, 4).



Required area = Area OMA

= Area OMP + Area MPA

$$= \int_0^4 x\cdot dx + \int_4^{4\sqrt{2}} \sqrt{(4\sqrt{2})^2 - x^2}\cdot dx$$

$$= \left[\frac{x^2}{2}\right]_0^4 + \left[\frac{x}{2}\sqrt{(4\sqrt{2})^2 - x^2} + \frac{(4\sqrt{2})^2}{2}\sin^{-1}\left(\frac{x}{4\sqrt{2}}\right)\right]_4^{4\sqrt{2}}$$

$$= \frac{16}{2} + \left[\left\{\frac{4\sqrt{2}}{2}\sqrt{(4\sqrt{2})^2 - (4\sqrt{2})^2} + \frac{32}{2}\sin^{-1}1\right\} - \left\{\frac{4}{2}\sqrt{(4\sqrt{2})^2 - (4)^2} + \frac{32}{2}\sin^{-1}\frac{1}{\sqrt{2}}\right\}\right]$$

$$= 8 + \left(2\sqrt{2}(0) + 16\times\frac{\pi}{2}\right) - \left(2\times 4 + 16\times\frac{\pi}{4}\right)$$

$$= 8 + 8\pi - 8 - 4\pi$$

$= 4\pi$ sq. units **Ans.**

27. Evaluate :

$$\int_0^{\pi/4} \frac{\sin x + \cos x}{16 + 9\sin 2x}dx$$

[6]

Solution :

Let $I = \int_0^{\pi/4} \frac{\sin x + \cos x}{16 + 9\sin 2x}.dx$

$$= \int_0^{\pi/4} \frac{\sin x + \cos x}{16 + 9[1 - (\sin x - \cos x)^2]}\cdot dx$$

$$\Rightarrow \quad = \int_0^{\pi/4} \frac{\sin x + \cos x}{25 - 9(\sin x - \cos x)^2}\cdot dx$$

Put $\sin x - \cos x = t$

$(\cos x + \sin x)\,dx = dt$

When $x = 0, t = -1$

and $x = \frac{\pi}{4}, t = 0$

$$I = \int_{-1}^0 \frac{dt}{25 - 9t^2} = \frac{1}{9}\int_{-1}^0 \frac{dt}{\frac{25}{9} - t^2}$$

$$= \frac{1}{9}\int_{-1}^0 \frac{dt}{\left(\frac{5}{3}\right)^2 - t^2}$$

$$= \frac{1}{9}\left[\frac{1}{2\times\frac{5}{3}}\log\left|\frac{\frac{5}{3}+t}{\frac{5}{3}-t}\right|\right]_{-1}^0$$

$$= \frac{1}{9}\left[\frac{1}{\frac{10}{3}}\left(\log\left|\frac{\frac{5}{3}}{\frac{5}{3}}\right| - \log\left|\frac{\frac{2}{3}}{\frac{8}{3}}\right|\right)\right]$$

$$= \frac{1}{30}\left[\log 1 - \log\frac{1}{4}\right]$$

$$= \frac{1}{30}[\log 1 - (\log 1 - \log 4)]$$

$$= \frac{1}{30}\log 4 = \frac{1}{30}\log(2)^2 = \frac{2}{30}\log 2$$

$= \frac{1}{15}\log 2$ **Ans.**

OR

Evaluate $\int_1^3 (x^2 + 3x + e^x)dx$ as the limit of the sum.

Solution : We have, $\int_1^3 (x^2 + 3x + e^x).dx$

Here, $f(x) = x^2 + 3x + e^x$, $a = 1$, $b = 3$, $nh = b - a = 3 - 1 = 2$

By limit of sum, we have

$$\int_a^b f(x)\,dx \lim_{h\to 0} h[f(a) + f(a+h) + f(a+2h) + ... + f(a + (n-1)h)]$$

where $nh = b - a$

$$\therefore \int_1^3 (x^2 + 3x + e^x).dx$$

$$= \lim_{h\to 0} h\,[f(1) + f(1+h) + f(2+h)\,..... + f(1 + \overline{n-1}h)]$$

$$= \lim_{h\to 0} h[(1 + 3 + e) + \{(1 + h^2) + 3(1 + h) + e^{1+h}\} + \{(1 + 2h)^2 + 3(1 + 2h) + e^{1+2h}\} + ...]$$

$$= \lim_{h\to 0} h(4 + e + (1 + h^2 + 2h + 3 + 3h + e^{1+h}) + (1 + 4h^2 + 4h + 3 + 6h + e^{1+2h}) + ...]$$

$$= \lim_{h\to 0} h\left[4 + e + (4 + h^2 + 5h + e^{1+h}) + (4 + 4h^2 + 10h + e^{1+2h}) + ...\right]$$

$$= \lim_{h\to 0} h[4n + e(1 + e^h + e^{2h} + ...) + h^2(1^2 + 2^2 + ...) + 5h(1 + 2 + ...)]$$

$$= \lim_{h\to 0} h\left[4n + e\left(\frac{e^{nh}-1}{e^h - 1}\right) + h^2\frac{n(n-1)(2n-1)}{6} + \frac{5hn(n-1)}{2}\right]$$

$$= \lim_{h \to 0}\left[4nh + e.\frac{h}{e^h - 1}(e^{nh} - 1) + \frac{nh(nh-h)(2nh-h)}{6} + \frac{5nh(nh-h)}{2}\right]$$

$$= 4(2) + e\cdot(e^2 - 1) + \frac{2(2-0)(4-0)}{6} + \frac{5(2)(2-0)}{2}$$

$$= 8 + (e^3 - e) + \frac{8}{3} + 10$$

$$= \frac{24+8+30}{3} + e^3 - e$$

$$= \frac{62}{3} + e^3 - e \qquad \textbf{Ans.}$$

28. Find the distance of the point (–1, –5, –10) from the point of intersection of the line $\vec{r} = 2\hat{i} - \hat{j} + 2\hat{k} + \lambda(3\hat{i} + 4\hat{j} + 2\hat{k})$ and the plane $\vec{r}.(\hat{i} - \hat{j} + \hat{k}) = 5$. [6]

Solution : Equation of line is

$$\vec{r} = 2\hat{i} - \hat{j} + 2\hat{k} + \lambda(3\hat{i} + 4\hat{j} + 2\hat{k}) \qquad ...(i)$$

Coordinates of any point on this line are

$$(2+3\lambda)\hat{i} + (-1+4\lambda)\hat{j} + (2+2\lambda)\hat{k}$$

Equation of plane is

$$\vec{r}.(\hat{i} - \hat{j} + \hat{k}) = 5 \qquad ...(ii)$$

Since, the point on line lies on the plane,

$$\vec{r}.(\hat{i} - \hat{j} + \hat{k}) = 5$$

$$[(2+3\lambda)\hat{i} + (2+4\lambda)\hat{j} + (2+2\lambda)](\hat{i} - \hat{j} + \hat{k}) = 5$$

$$\Rightarrow (2+3\lambda) - (-1+4\lambda) + (2+2\lambda) = 5$$

$$\Rightarrow \lambda + 5 = 5$$

$$\Rightarrow \lambda = 0$$

So equation of line is

$$\vec{r} = (2\hat{i} - \hat{j} + 2\hat{k}) + 0(3\hat{i} + 4\hat{j} + 2\hat{k})$$

$$\Rightarrow \vec{r} = 2\hat{i} - \hat{j} + 2\hat{k} \qquad ...(iii)$$

Let, point of intersection be (x, y, z)

$$\therefore \vec{r} = x\hat{i} + y\hat{j} + z\hat{k} \qquad ...(iv)$$

From equations (iii) and (iv)

$$x = 2, y = -1, z = 2$$

$\therefore$ Point of intersection is (2, –1, 2).
Distance between points (2, –1, 2) and (–1, –5, –10) is

$$= \sqrt{(-1-2)^2 + (-5+1)^2 + (-10-2)^2}$$

$$= \sqrt{(-3)^2 + (-4)^2 + (-12)^2}$$

$$= \sqrt{9+16+144} = \sqrt{169} = 13 \text{ units} \qquad \textbf{Ans.}$$

29. A factory manufactures two types of screws A and B, each type requiring the use of two machines, an automatic and a hand-operated. It takes 4 minutes on the automatic and 6 minutes on the hand-operated machines to manufacture a packet of screws 'A' while it takes 6 minutes on the automatic and 3 minutes on the hand-operated machine to manufacture a packet of screws 'B'. Each machine is available for at most 4 hours on any day. The manufacturer can sell a packet of screws 'A' at a profit of 70 paise and screws 'B' at a profit of ₹1. Assuming that he can sell all the screws he manufactures, how many packets of each type should the factory owner produce in a day in order to maximize his profit ? Formulate the above LPP and solve it graphically and find the maximum profit. [6]

Solution : Let the number of packets of screw 'A' manufactured in a day be x and that of screw B be y.

Therefore, $x \geq 0, y \geq 0$

Item	Number	Machine A	Machine B	Profit
Screw A	x	4 minutes	6 minutes	₹ 0.7
Screw B	y	6 minutes	3 minutes	₹ 1
Max. time available		4 hrs. = 240 min.	4 hrs. = 240 min.	

Then, the constraints are :

$$4x + 6y \leq 240 \text{ or } 2x + 3y \leq 120$$
$$6x + 3y \leq 240 \text{ or } 2x + y \leq 80$$

and total profit,

$$Z = 0.7x + y$$

So our LPP will be

$$\text{Max. } Z = 0.7x + y$$

Subject to the constraints :

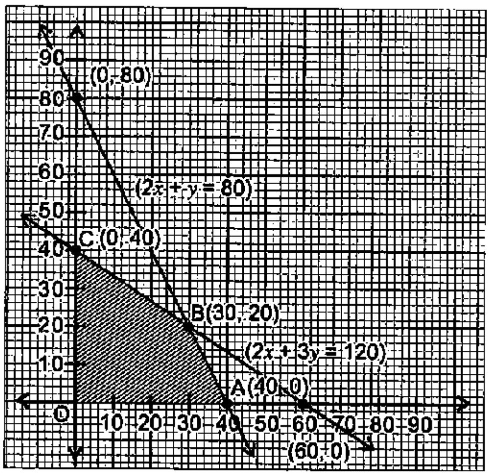
$$2x + 3y \leq 120,$$
$$2x + y \leq 80,$$
$$\text{and} \quad x, y \geq 0$$

Now, $2x + 3y \leq 120$ and $2x + y \leq 80$

x	0	60
y	40	0

x	0	40
y	80	0

Plotting the points on the graph, we get the feasible region OABC as shown (Shaded).



Corner points	Value of Z = 0.7x + y
C(0, 40)	0.7(0) + 40 = 40
B (30, 20)	0.7 (30) + 20 = 41 Maximum
A(40, 0)	0.7(40) + 0 = 28
O (0, 0)	0.7 (0) + 0 = 0

Hence, profit will be maximum if company produces 30 packets of screw A and 20 packets of screw B and maximum profit = ₹ 41. **Ans.**

Mathematics 2017 (Outside Delhi) SET I

Time allowed : 3 hours **Maximum marks : 100**

SECTION — A

1. If for any 2 × 2 square matrix A,

A(adj A) = $\begin{bmatrix} 8 & 0 \\ 0 & 8 \end{bmatrix}$**,**

then write the value of | A |. **[1]**

Solution : We have,

$$A(\text{adj } A) = \begin{bmatrix} 8 & 0 \\ 0 & 8 \end{bmatrix}$$

As, $A(\text{adj } A) = |A|\, I$

$$\begin{bmatrix} 8 & 0 \\ 0 & 8 \end{bmatrix} = 8\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

On comparing, we get

$$|A| = 8.$$ **Ans.**

2. Determine the value of 'k' for which the following function is continuous at $x = 3$.

$$f(x) = \begin{cases} \dfrac{(x+3)^2 - 36}{x-3}, & x \neq 3 \\ k, & x = 3 \end{cases}$$ **[1]**

Solution : Given,

$$f(x) = \begin{cases} \dfrac{(x+3)^2 - 36}{x-3}, & x \neq 3 \\ k, & x = 3 \end{cases}$$

Since $f(x)$ is continuous at $x = 3$

$\therefore \quad \lim_{x \to 3} f(x) = f(3)$

$\Rightarrow \quad \lim_{x \to 3} f(x) = k$

$\Rightarrow \quad \lim_{x \to 3} \dfrac{(x+3)^2 - 36}{x-3} = k$

$\Rightarrow \quad \lim_{x+3 \to 6} \dfrac{(x+3)^2 - 6^2}{(x+3) - 6} = k$

$\Rightarrow \quad \lim_{x \to 3} x + 3 + 6 = k$

$\Rightarrow \quad 12 = k$

Thus, $f(x)$ is continuous at $x = 3$; if $k = 12$. **Ans.**

3. Find : $\displaystyle\int \frac{\sin^2 x - \cos^2 x}{\sin x \cos x}\, dx.$ **[1]**

Solution : We have,

$$\int \frac{\sin^2 x - \cos^2 x}{\sin x \cos x}\, dx = -2\int \frac{\cos^2 x - \sin^2 x}{2 \sin x \cos x}\, dx$$

$$= -2\int \frac{\cos 2x}{\sin 2x}\, dx$$

$$= -2\int \cot 2x \, dx$$

$$= -\log |\sin 2x| + C$$

Ans.

4. Find the distance between the planes $2x - y + 2z = 5$ and $5x - 2.5y + 5z = 20$. **[1]**